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On linear-algebraic notions of expansion

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Note
In this poster, we employ the RED color to highlight our new results.

Graph-theoretic Expansion

Classical expanders are sparse graphs in high connectivity that have wide applications. We briefly recall the following three notions of expansion defined from different perspectives.

Let $G = ([n], E)$ be a d-regular graph.

• The *spectral expansion* of G, $\lambda(G)$, is defined as the second-largest absolute value over all eigenvalues of A_G, where A_G is the adjacency matrix of G.

• The *edge expansion* of G, $h(G)$, is defined as

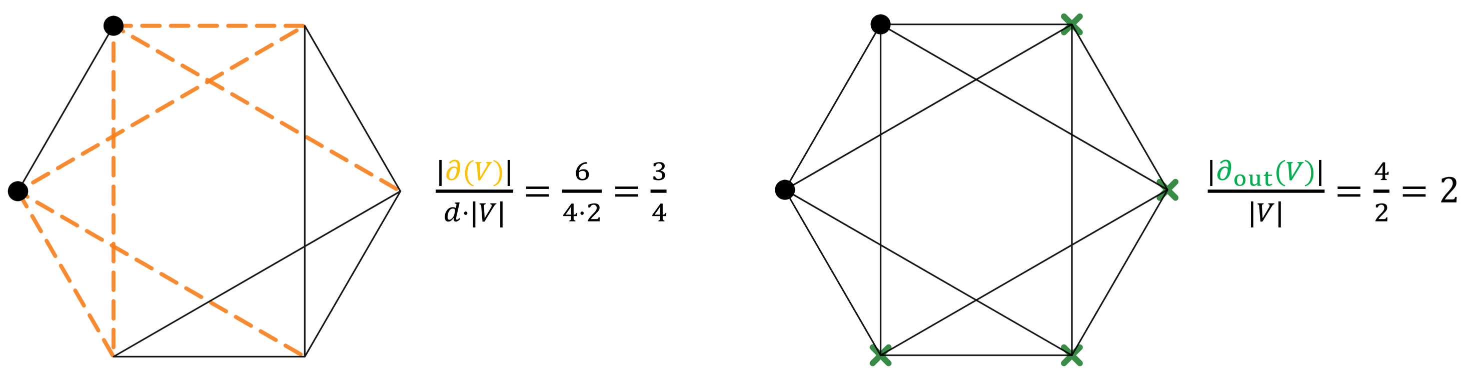
$$h(G) := \min_{\substack{V\subseteq[n]\\ 1\le |W|\le \frac{n}{2}}} \frac{|\partial(V)|}{d\cdot|V|},$$

where $\partial(V) := \{\{i,j\}\in E : i\in V, j\in[n]\setminus V\}$.

• The *vertex expansion* of G, $\mu(G)$, is defined as

$$\mu(G) := \min_{\substack{V\subseteq[n]\\ 1\le |W|\le \frac{n}{2}}} \frac{|\partial_{\text{out}}(V)|}{|V|},$$

where $\partial_{\text{out}}(V) := \{j\in[n]\setminus V : \exists i\in V, \text{ s.t. } \{i,j\}\in E\}$.



Classical results [2,3,4]:

For any d-regular graph G, the above three notions of expansion are all **equivalent** in the sense that

(1) $\frac{\mu(G)}{d} \le h(G) \le \mu(G)$;

(2) $\frac{1-\lambda(G)}{2} \le h(G) \le \sqrt{2(1-\lambda(G))}$.

Linear-algebraic Expansion

There are several well-studied notions of linear-algebraic expansion, including quantum expansion [5, 6], quantum edge expansion [7] and dimension expansion [8], defined in natural analogy to the left-three graph-theoretic notions, respectively. In our paper, we also defined a new notion termed dimension edge expansion [1].

We mainly work with *doubly stochastic matrix tuples*, which are those matrix tuples $\mathbf{B} = (B_1, \dots, B_d) \in \mathrm{M}(n,\mathbb{C})^d$ with $\sum_{i=1}^d B_i B_i^* = \sum_{i=1}^d B_i^* B_i = dI_n$. The *associated quantum operator* is the linear map $\Phi_{\mathbf{B}}\colon \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(n,\mathbb{C})$ defined by

$$\Phi_{\mathbf{B}}(X) := \frac{1}{d}\sum_{i=1}^{d} B_i X B_i^* .$$

• The *quantum expansion* of $\Phi_{\mathbf{B}}$, $\lambda(\mathbf{B})$, is defined as the second-largest absolute value over all eigenvalues of $\Phi_{\mathbf{B}}$.

• The *quantum edge expansion* of $\Phi_{\mathbf{B}}$, $h_Q(\mathbf{B})$, is defined as

$$h_Q(\mathbf{B}) := \min_{\substack{V\le\mathbb{C}^n\\ 1\le \dim(V)\le \frac{n}{2}}} \frac{\langle I_n - P_V, \Phi_{\mathbf{B}}(P_V)\rangle}{\dim(V)},$$

where P_V is the orthogonal projection to the subspace $V \le \mathbb{C}^n$, and $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathrm{M}(n,\mathbb{C})$.

• The *dimension expansion* of $\mathbf{B}$, $\mu(\mathbf{B})$, is defined as

$$\mu(\mathbf{B}) := \min_{\substack{V\le\mathbb{C}^n\\ 1\le \dim(V)\le \frac{n}{2}}} \frac{\dim(V+\mathbf{B}(V)) - \dim(V)}{\dim(V)},$$

where $\mathbf{B}(V) := \langle \cup_{i\in[d]} \{B_i v : v\in V\}\rangle$ and $\langle\cdot\rangle$ denotes the linear span over $\mathbb{C}$.

• The *dimension edge expansion* of $\mathbf{B}$, $h_D(\mathbf{B})$, is defined as

$$h_D(\mathbf{B}) := \min_{\substack{V\le\mathbb{C}^n\\ 1\le \dim(V)\le \frac{n}{2}}} \frac{\sum_{i=1}^d \operatorname{rank}(T_{V^\perp}^* B_i T_V)}{d\cdot \dim(V)},$$

where $T_V \in \mathrm{M}(n\times \dim V, \mathbb{C})$ is a matrix whose columns form an orthonormal basis of V, and $T_{V^\perp} \in \mathrm{M}(n\times \dim V^\perp, \mathbb{C})$ is a matrix whose columns form an orthonormal basis of $V^\perp$, the orthogonal complement of V.

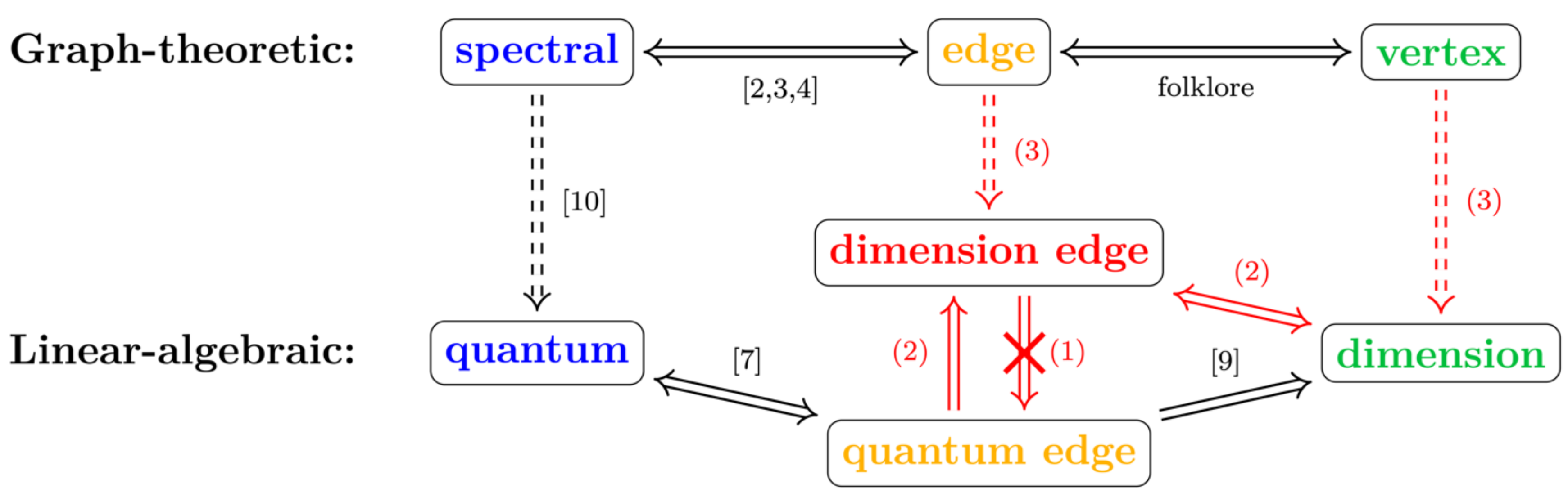
Quick example:

$$B(V) = \begin{bmatrix} * & * & * \\ * & * & * \\ \mathbf{b}_{31} & \mathbf{b}_{32} & * \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \\ \mathbf{b}_{31} & \mathbf{b}_{32} \end{bmatrix}$$

$$T_{V^\perp}^{t} B T_V = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^{t} \cdot \begin{bmatrix} * & * & * \\ * & * & * \\ \mathbf{b}_{31} & \mathbf{b}_{32} & * \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = [\mathbf{b}_{31} \quad \mathbf{b}_{32}]$$

$$\dim(B(V)+V) - \dim(V) = \operatorname{rank}(T_{V^\perp}^{t} B T_V)$$

Overview



Remark: (1), (2) and (3) refer to our three main results, [2,3,4], [7], [9] and [10] refer to the corresponding references in the last section. The solid arrows indicate the implication or equivalence, while the dashed arrows represent proper generalizations.

(1) Dimension expansion does not imply quantum expansion: We showed the existence of dimension expanders that are arbitrarily poor quantum expanders (and thus arbitrarily poor quantum edge expanders).

(2) Quantum expansion implies dimension expansion: We showed that any quantum expander based on an unital quantum channel gives rise to a dimension expander, via our new notion of dimension edge expansion. This also leads to a new and more modular proof of Lubotzky-Zelmanov result [9] with a stronger bound.

(3) Dimension expansion and dimension edge expansion are indeed proper generalizations of vertex expansion and edge expansion, respectively: We showed a quantitative equivalence in the sense of graphical matrix tuples. In another paper [11], we also proved that many important graph-theoretic properties are equivalent to linear-algebraic properties of the associated graphical matrix tuple.

Main Result (1)

We proved that **there exist dimension expanders which have arbitrarily poor quantum expansion**.

Specifically, we showed that for any matrix tuple $\mathbf{B} = (B_1, \dots, B_d) \in \mathrm{M}(n,\mathbb{C})^d$ consisting of unitary matrices, $\mathbf{B}^s := (B_1^s, \dots, B_d^s) \in \mathrm{M}(n,\mathbb{C})^d$ satisfies that

$$\mu(\mathbf{B}^s) \ge \mu(\mathbf{B})/d \text{ and } \lambda(\mathbf{B}) < \varepsilon$$

for all $\varepsilon > 0$, all sufficiently large $n \in \mathbb{N}$, and some sufficiently small power $s > 0$. This proof requires some more advanced techniques, such as a number of compactness arguments.

We also showed that there exist dimension expanders $\mathbf{B}$ consisting of unitary matrices such that for all $s > 0$, $\mathbf{B}^s$ is not a quantum expander. It concludes that we cannot convert any dimension expander into one that is also a quantum expander by taking a large power s.

Main Result (2)

We proved that **every unital quantum expander is a dimension expander** by

$$\frac{1-\lambda(\mathbf{B})}{2d} \le \frac{h_Q(\mathbf{B})}{d} \le h_D(\mathbf{B}) \le \mu(\mathbf{B}),$$

where the first inequality was given by Hastings [7]. It follows that if there is a spectral gap $1-\lambda(\mathbf{B}) > 0$, then $\mu(\mathbf{B}) > 0$.

In case $\mathbf{B}$ consists of unitary matrices only, we can get rid of d to make a stronger bound, i.e.,

$$\frac{1-\lambda(\mathbf{B})}{2} \le h_Q(\mathbf{B}) \le h_D(\mathbf{B}) \le \mu(\mathbf{B}).$$

This improves the result of Lubotzky and Zelmanov [9] where in the same setting they proved

$$\frac{1-\lambda(\mathbf{B})}{6} \le \mu(\mathbf{B}).$$

Main Result (3)

We studied the *graphical matrix tuple* $\mathbf{B}_G := (\sqrt{n}\cdot \mathrm{E}_{i,j} : \{i,j\}\in E)$ associated to a d-regular graph $G = ([n], E)$, where $\mathrm{E}_{i,j}$ is the *elementary matrix* with a 1 in position (i,j) and zeros in all other entries. For example, when $n = 3$,

$$\mathrm{E}_{2,3} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Inspired by a result of Bannink, Briët, Labib, and Maassen [10] who proved for any d-regular graph G,

$$\lambda(\mathbf{B}_G) = \lambda(G),$$

we showed some analogous results that for any d-regular graph G,

(1) $h_Q(\mathbf{B}_G) \not\equiv h(G)$;

(2) $h_D(\mathbf{B}_G) = h(G)$;

(3) $\mu(\mathbf{B}_G) = \mu(G)$.

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