Diffie–Hellman Key Exchange from Commutativity to Group Laws

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This work has been accepted to ITCS 2026. For more details, please refer to our preprint at https://eprint.iacr.org/2025/1677.pdf

Outline

- Review the classical Diffie–Hellman key exchange.

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- Propose our group action-based key exchange framework.

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- Propose our group action-based key exchange framework.
- Instantiate the framework by linear code equivalence problems.

What is key exchange?

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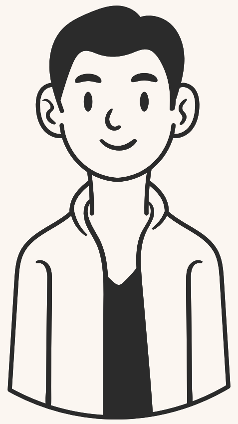
  - The shared secret is computed from the combination of a public key and one's private key.
  - An adversary can eavesdrop on all transmitted messages.
- Application: HTTPS, VPN, and messaging services.

Diffie–Hellman key exchange protocol

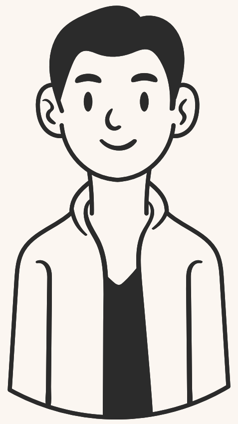
Alice



Bob

Diffie–Hellman key exchange protocol

pk : prime p and generator γ of a cyclic group C_p



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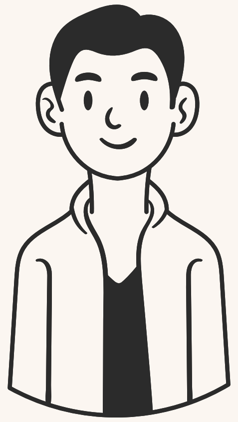

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

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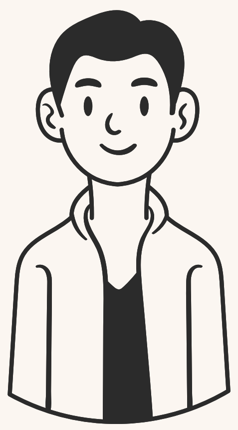

Alice

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$a \xleftarrow{\$} \mathbb{Z}_p^*$

$A = \gamma^a$

Diffie–Hellman key exchange protocol

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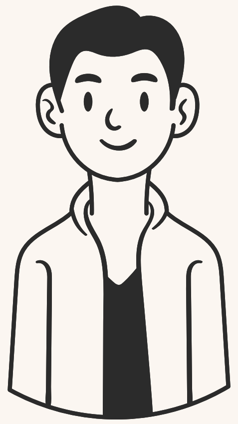

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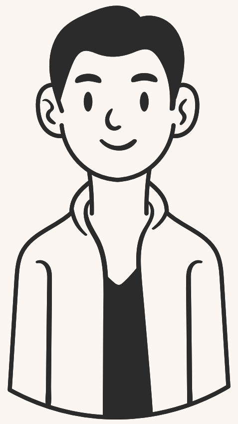

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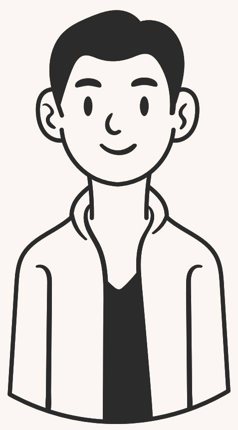

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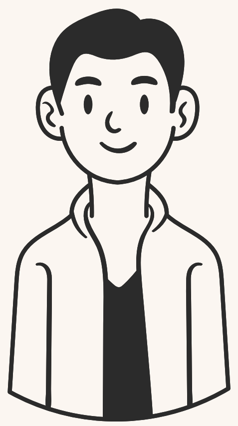

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$A = \gamma^a$

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$$\xrightarrow{\quad A \quad}$$

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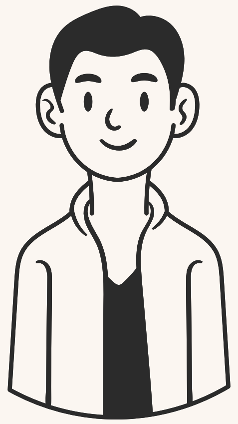

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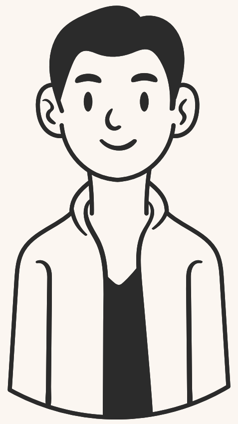

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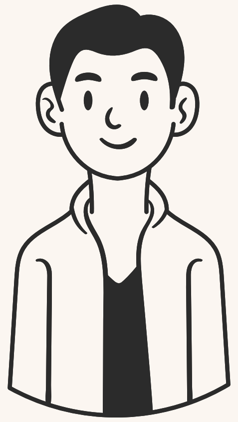

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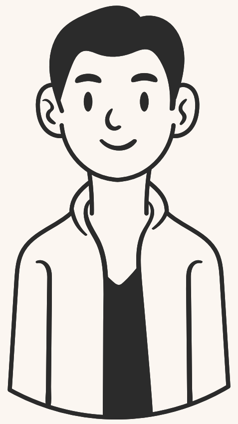

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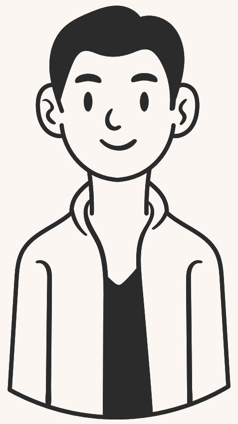

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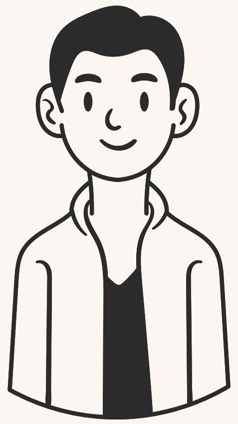

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$a \xleftarrow{\$} \mathbb{Z}_p^*$	Given γ, γ^a, it's hard to solve a!	$b \xleftarrow{\$} \mathbb{Z}_p^*$
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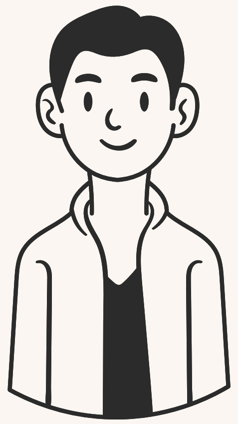

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Discrete Log Assumption

Given γ, γ^a, it's hard to solve a!

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$A = \gamma^a$		$B = \gamma^b$
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	$\xleftarrow{\quad B \quad}$	
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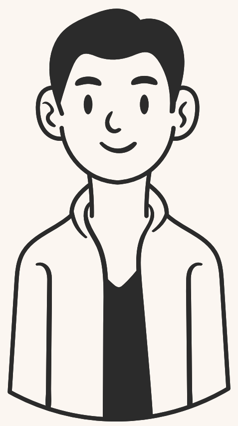
Correctness: $A^b = \gamma^{ab} = \gamma^{ba} = B^a$.

Diffie–Hellman key exchange protocol from group action view

Alice



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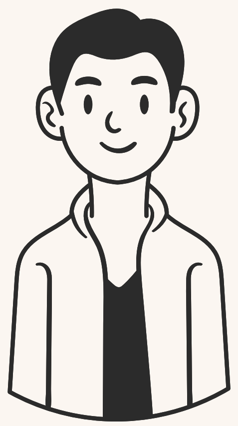
$\mathsf{pk} : s \in S$



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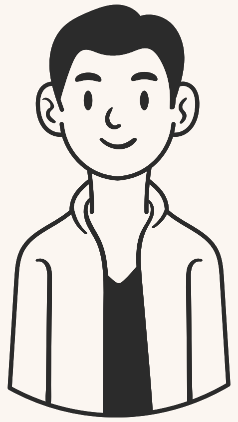

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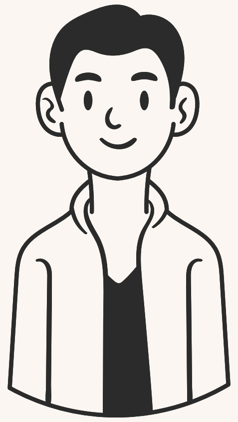

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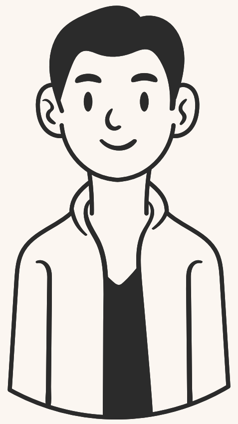

Bob

$g \overset{\$}{\leftarrow} G$

$A = s * g$

$h \overset{\$}{\leftarrow} G$

$B = s * h$

$$\xrightarrow{\quad A \quad}$$

$$\xleftarrow{\quad B \quad}$$

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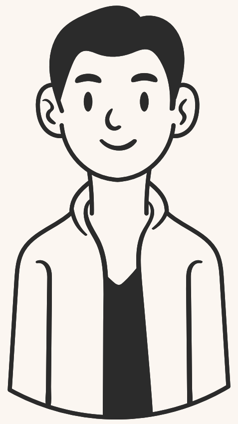

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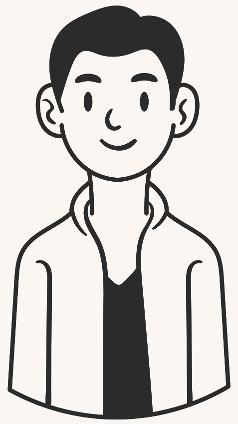

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Correctness: $B * g = s * hg = s * gh = A * h.$

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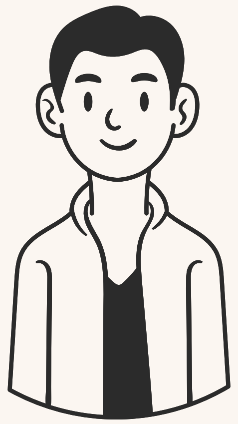

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$$\xrightarrow{\quad A \quad}$$

$$\xleftarrow{\quad B \quad}$$

$\mathbf{key} = B * g$ $\qquad\qquad\qquad$ $\mathbf{key} = A * h$

Correctness: $B * g = s * \color{red}{hg} = s * \color{red}{gh} = A * h.$

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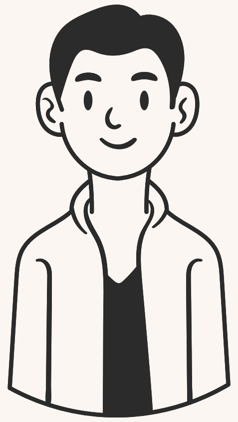

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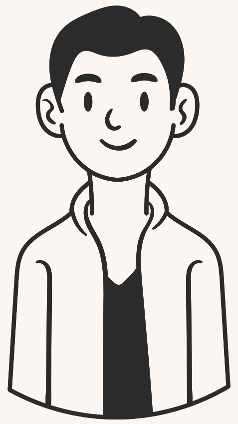

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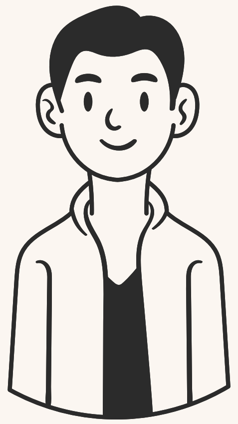

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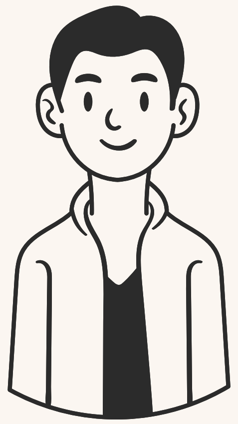

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One-way Hardness Assumption[1]

Given s, $s * g$ it's hard to solve g!

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[1] [Brassard-Yung, *Crypto*, 90]

Diffie–Hellman key exchange protocol from group action view

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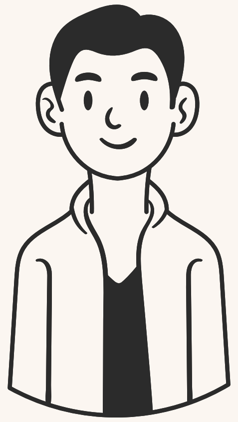

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Good candidate for instantiation

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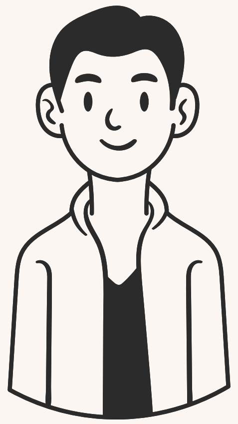

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$S = C_p \setminus \mathrm{id}$

$G = \mathrm{Aut}(C_p)$

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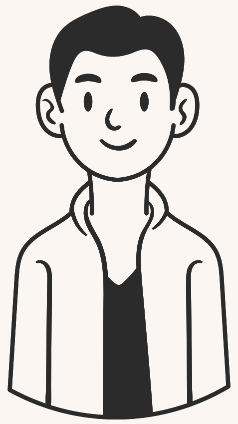

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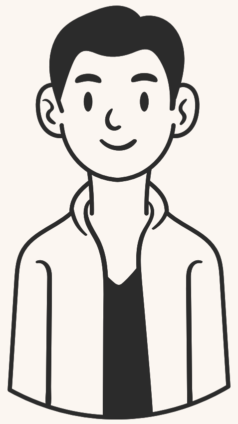

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$\boxed{\mathrm{Aut}(C_p) \cong \mathbb{Z}_p^*}$

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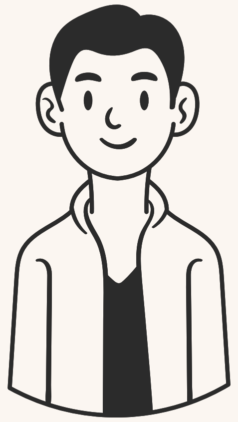

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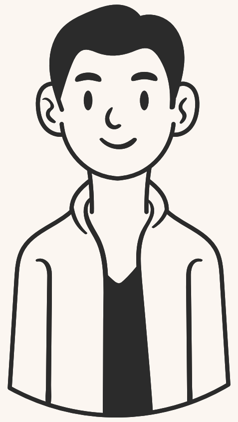

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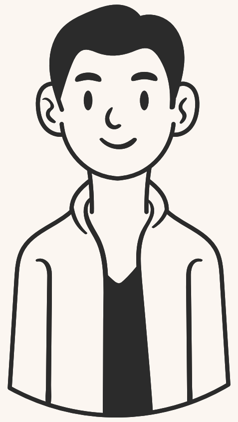
Correctness: $B * g = s * hg = s * gh = A * h.$

Beyond commutativity?

Diffie–Hellman key exchange protocol from group action view

Alice

$\mathsf{pk} : s \in S$



Bob

Alice		Bob
$g \xleftarrow{\$} G$		$h \xleftarrow{\$} G$
$A = s * g$	$\xrightarrow{\quad A \quad}$	$B = s * h$
$\mathbf{key} = B * g$	$\xleftarrow{\quad B \quad}$	$\mathbf{key} = A * h$

Correctness: $B * g = s * {\color{red}hg} = s * {\color{red}gh} = A * h.$

Idea: treating this as a law in a group!

Law in a group

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A *law* in a group G is an equation that is satisfied by any assignments of variables by group elements in G.

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- $aba^{-1}b^{-1}cdc^{-1}d^{-1} = cdc^{-1}d^{-1}aba^{-1}b^{-1}$ is a law in a metabelian group.
  $[a, b], [c, d] = [c, d][a, b]$

Law in a group

A *law* in a group G is an equation that is satisfied by any assignments of variables by group elements in G.

- $ab = ba$ is a law in an abelian group.

- $aba^{-1}b^{-1}cdc^{-1}d^{-1} = cdc^{-1}d^{-1}aba^{-1}b^{-1}$ is a law in a metabelian group.

- $u(a, b, c, \ldots) = v(a, b, c, \ldots)$ is a law in a group.

Law in a group

A *law* in a group G is an equation that is satisfied by any assignments of variables by group elements in G.

- $ab = ba$ is a law in an abelian group.

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- $\underset{\substack{\downarrow \\ \text{word}}}{u}(a, b, c, \ldots) = v(a, b, c, \ldots)$ is a law in a group.

Law in a group

A *law* in a group G is an equation that is satisfied by any assignments of variables by group elements in G.

- $ab = ba$ is a law in an abelian group.
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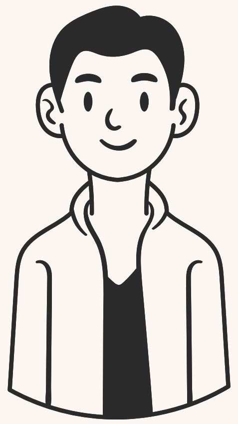
- $u(a, b, c, \ldots) = v(a, b, c, \ldots)$ is a law in a group.

word: e.g., $a^2b^3a^{-5}c^2b^7$

Key exchange protocol for actions of groups with laws

Alice

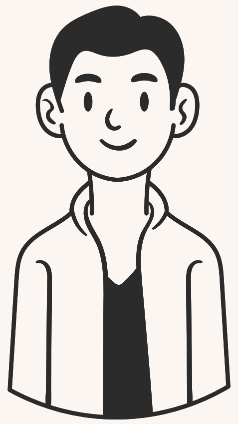


Bob

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

Key exchange protocol for actions of groups with laws

Alice

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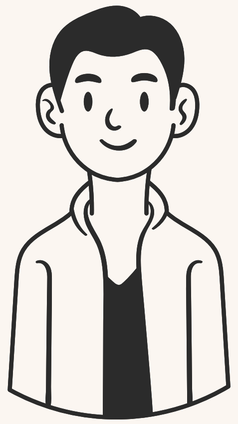


Bob

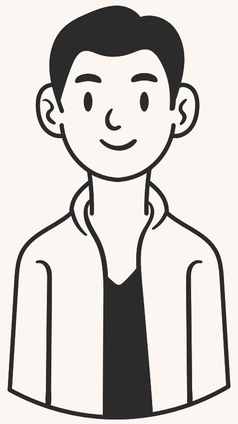
$$y^{b_1} x^{a_1} \ldots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \ldots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

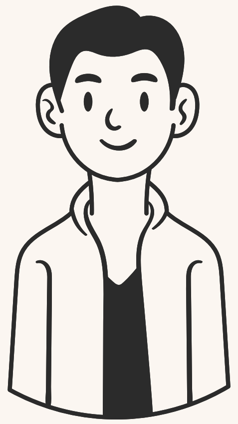
$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \ldots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \ldots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$g \xleftarrow{\$} G$

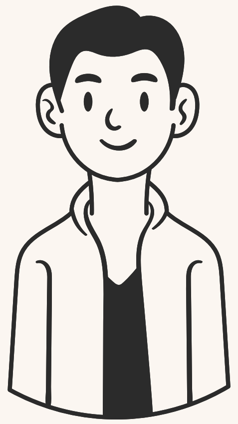
$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



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for any $x, y \in G$

$g \xleftarrow{\$} G$

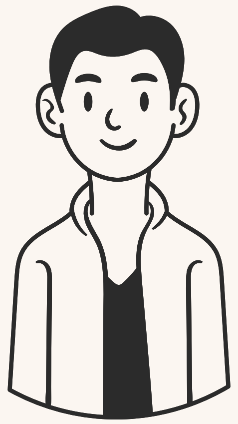
$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

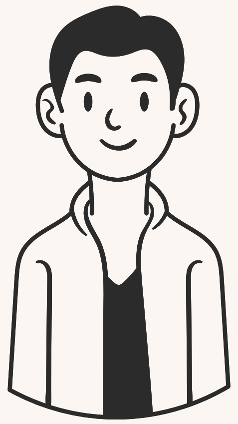
$\xleftarrow{\quad t_1 \quad}$

$s_1 = t_1 * g^{a_1}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \ldots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \ldots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

$s_1 = t_1 * g^{a_1}$

$\xrightarrow{\quad s_1 \quad}$

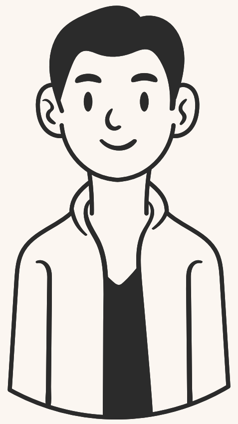
Key exchange protocol for actions of groups with laws

Alice

$$\mathsf{pk} : s_0 \in S$$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$



Bob

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$$\xleftarrow{\quad t_1 \quad}$$

$s_1 = t_1 * g^{a_1}$

$$\xrightarrow{\quad s_1 \quad}$$

$t_2 = s_1 * h^{b_2}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
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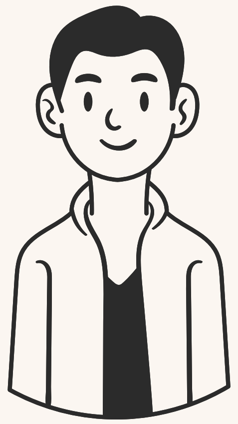
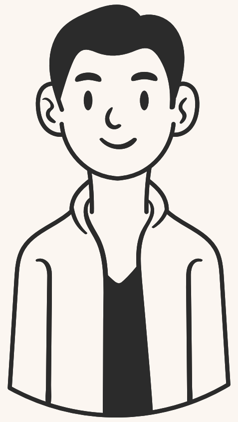


Bob

Alice		Bob
$g \xleftarrow{\$} G$		$h \xleftarrow{\$} G$
	$\xleftarrow{\quad t_1 \quad}$	$t_1 = s_0 * h^{b_1}$
$s_1 = t_1 * g^{a_1}$	$\xrightarrow{\quad s_1 \quad}$	
	$\xleftarrow{\quad t_2 \quad}$	$t_2 = s_1 * h^{b_2}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

Alice: $g \xleftarrow{\$} G$

Bob: $h \xleftarrow{\$} G$

Bob: $t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

Alice: $s_1 = t_1 * g^{a_1}$

$\xrightarrow{\quad s_1 \quad}$

Bob: $t_2 = s_1 * h^{b_2}$

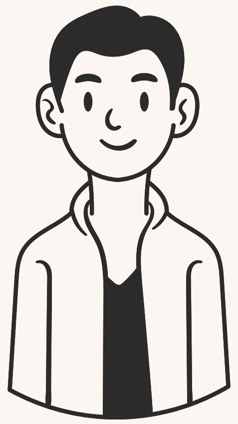
$\xleftarrow{\quad t_2 \quad}$

Alice: $s_2 = t_2 * g^{a_2}$

Key exchange protocol for actions of groups with laws

$\mathsf{pk} : s_0 \in S$

Alice



Bob

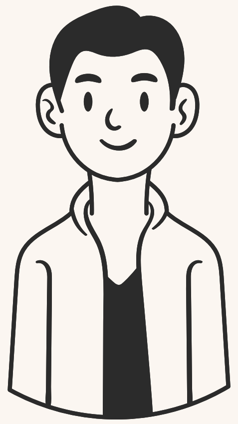
$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

Alice		Bob
$g \xleftarrow{\$} G$		$h \xleftarrow{\$} G$
	$\xleftarrow{t_1}$	$t_1 = s_0 * h^{b_1}$
$s_1 = t_1 * g^{a_1}$	$\xrightarrow{s_1}$	
	$\xleftarrow{t_2}$	$t_2 = s_1 * h^{b_2}$
$s_2 = t_2 * g^{a_2}$		
	$\vdots$	

Key exchange protocol for actions of groups with laws

Alice

pk : $s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

$s_1 = t_1 * g^{a_1}$

$\xrightarrow{\quad s_1 \quad}$

$t_2 = s_1 * h^{b_2}$

$\xleftarrow{\quad t_2 \quad}$

$s_2 = t_2 * g^{a_2}$

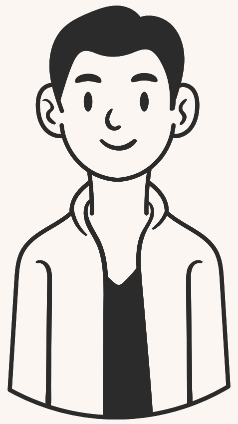
$\vdots$

$t_k = s_{k-1} * h^{b_k}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

$s_1 = t_1 * g^{a_1}$

$\xrightarrow{\quad s_1 \quad}$

$t_2 = s_1 * h^{b_2}$

$\xleftarrow{\quad t_2 \quad}$

$s_2 = t_2 * g^{a_2}$

$\vdots$

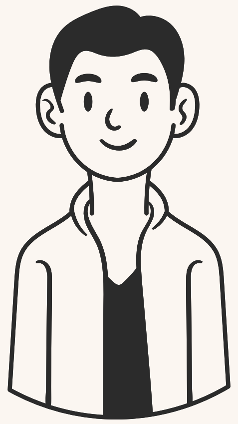
$t_k = s_{k-1} * h^{b_k}$

$\xleftarrow{\quad t_k \quad}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \ldots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \ldots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

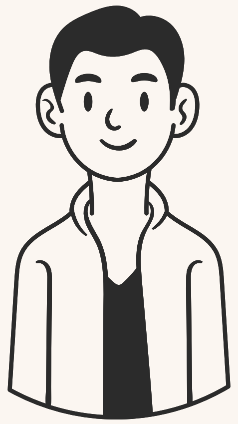
Alice		Bob
$g \xleftarrow{\$} G$		$h \xleftarrow{\$} G$
	$\xleftarrow{\quad t_1 \quad}$	$t_1 = s_0 * h^{b_1}$
$s_1 = t_1 * g^{a_1}$	$\xrightarrow{\quad s_1 \quad}$	
	$\xleftarrow{\quad t_2 \quad}$	$t_2 = s_1 * h^{b_2}$
$s_2 = t_2 * g^{a_2}$		
	$\vdots$	
	$\xleftarrow{\quad t_k \quad}$	$t_k = s_{k-1} * h^{b_k}$

key : $s_k = t_k * g^{a_k}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$t_1 = s_0 * h^{b_1}$

$\xleftarrow{\quad t_1 \quad}$

$s_1 = t_1 * g^{a_1}$

$\xrightarrow{\quad s_1 \quad}$

$t_2 = s_1 * h^{b_2}$

$\xleftarrow{\quad t_2 \quad}$

$s_2 = t_2 * g^{a_2}$

$\vdots$

$t_k = s_{k-1} * h^{b_k}$

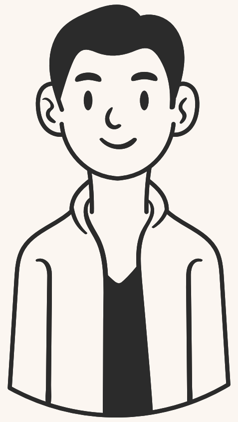
$\xleftarrow{\quad t_k \quad}$

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1}g^{a_1}\dots h^{b_k}g^{a_k}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

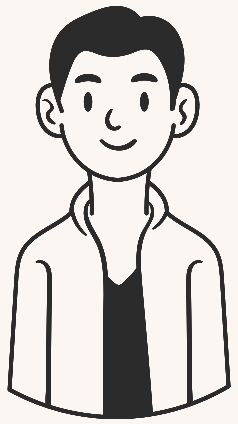
for any $x, y \in G$

$s_1' = s_0 * g^{c_1}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

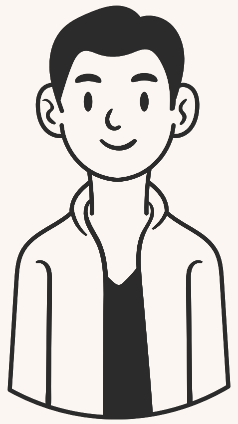
for any $x, y \in G$

$s_1' = s_0 * g^{c_1}$

$\xrightarrow{s_1'}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk}: s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

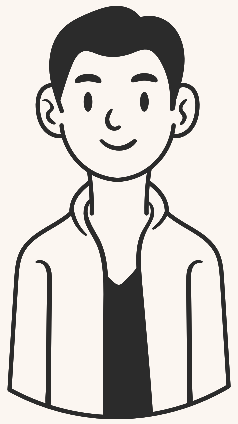
$s_1' = s_0 * g^{c_1}$

$\xrightarrow{\;s_1'\;}$

$t_1' = s_1' * h^{d_1}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$s'_1 = s_0 * g^{c_1}$

$$\xrightarrow{\quad s'_1 \quad}$$

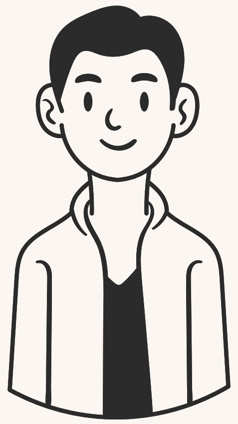
$$\xleftarrow{\quad t'_1 \quad}$$

$t'_1 = s'_1 * h^{d_1}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$s_1' = s_0 * g^{c_1}$

$\xrightarrow{\quad s_1' \quad}$

$t_1' = s_1' * h^{d_1}$

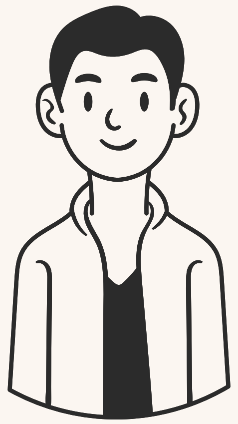
$\xleftarrow{\quad t_1' \quad}$

$s_2' = t_1' * g^{c_2}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

$s_1' = s_0 * g^{c_1}$

$\xrightarrow{s_1'}$

$\xleftarrow{t_1'}$

$t_1' = s_1' * h^{d_1}$

$s_2' = t_1' * g^{c_2}$

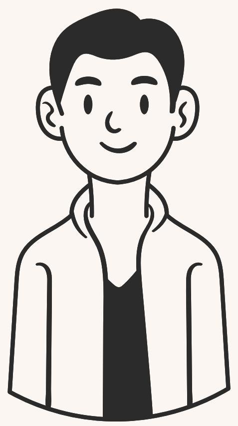
$\xrightarrow{s_2'}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$



Bob

$s_1' = s_0 * g^{c_1}$

$$\xrightarrow{\quad s_1' \quad}$$

$t_1' = s_1' * h^{d_1}$

$$\xleftarrow{\quad t_1' \quad}$$

$s_2' = t_1' * g^{c_2}$

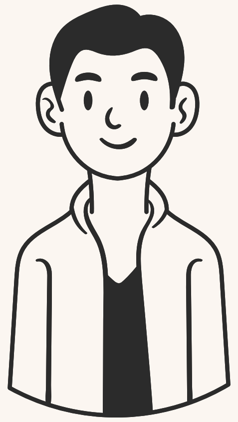
$$\xrightarrow{\quad s_2' \quad}$$

$t_2' = s_2' * h^{d_2}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$s'_1 = s_0 * g^{c_1}$

$$\xrightarrow{\quad s'_1 \quad}$$

$t'_1 = s'_1 * h^{d_1}$

$$\xleftarrow{\quad t'_1 \quad}$$

$s'_2 = t'_1 * g^{c_2}$

$$\xrightarrow{\quad s'_2 \quad}$$

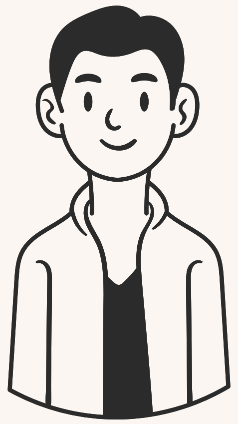
$t'_2 = s'_2 * h^{d_2}$

$$\vdots$$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

$s'_1 = s_0 * g^{c_1}$

$\xrightarrow{\quad s'_1 \quad}$

$t'_1 = s'_1 * h^{d_1}$

$\xleftarrow{\quad t'_1 \quad}$

$s'_2 = t'_1 * g^{c_2}$

$\xrightarrow{\quad s'_2 \quad}$

$t'_2 = s'_2 * h^{d_2}$

$\vdots$

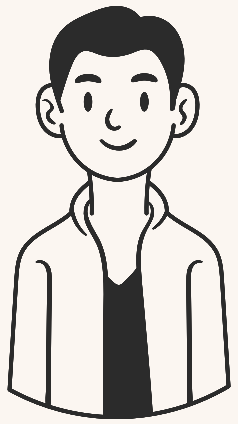
$s'_\ell = t'_{\ell-1} * g^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$



Bob

$s_1' = s_0 * g^{c_1}$ $\xrightarrow{s_1'}$

$\xleftarrow{t_1'}$ $t_1' = s_1' * h^{d_1}$

$s_2' = t_1' * g^{c_2}$ $\xrightarrow{s_2'}$

$t_2' = s_2' * h^{d_2}$

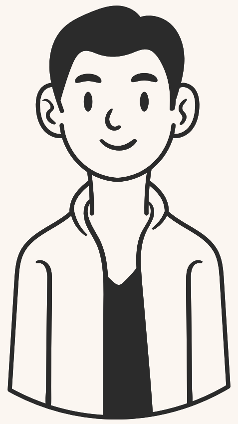
$\vdots$

$s_\ell' = t_{\ell-1}' * g^{d_\ell}$ $\xrightarrow{s_\ell'}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

$s_1' = s_0 * g^{c_1}$

$$\xrightarrow{\quad s_1' \quad}$$

$t_1' = s_1' * h^{d_1}$

$$\xleftarrow{\quad t_1' \quad}$$

$s_2' = t_1' * g^{c_2}$

$$\xrightarrow{\quad s_2' \quad}$$

$t_2' = s_2' * h^{d_2}$

$\vdots$

$s_\ell' = t_{\ell-1}' * g^{d_\ell}$

$$\xrightarrow{\quad s_\ell' \quad}$$

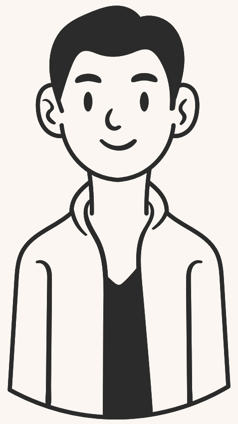
key : $t_\ell' = s_\ell' * h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = {\color{red}x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}}$$
for any $x, y \in G$



Bob

$s_1' = s_0 * g^{c_1}$

$\xrightarrow{\quad s_1' \quad}$

$t_1' = s_1' * h^{d_1}$

$\xleftarrow{\quad t_1' \quad}$

$s_2' = t_1' * g^{c_2}$

$\xrightarrow{\quad s_2' \quad}$

$t_2' = s_2' * h^{d_2}$

$\vdots$

$s_\ell' = t_{\ell-1}' * g^{d_\ell}$

$\xrightarrow{\quad s_\ell' \quad}$

$\mathbf{key} : t_\ell' = s_\ell' * h^{d_\ell} = s_0 * {\color{red}g^{c_1}h^{d_1}\dots g^{c_\ell}h^{d_\ell}}$

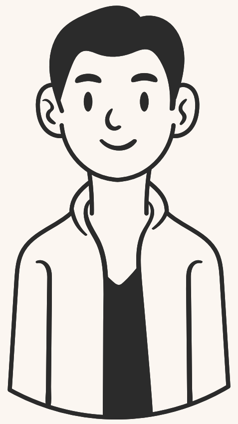
Key exchange protocol for actions of groups with laws

Alice

$g \xleftarrow{\$} G$

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$



Bob

$h \xleftarrow{\$} G$

$\longrightarrow$

$\longleftarrow$

$\longrightarrow$

$\vdots$

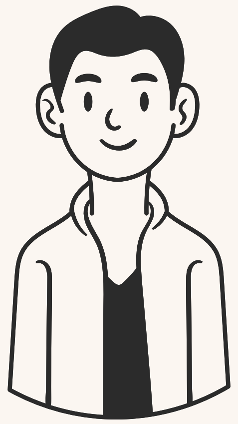
$\vdots$

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1} g^{a_1} \dots h^{b_k} g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1} h^{d_1} \dots g^{c_\ell} h^{d_\ell}$

Key exchange protocol for actions of groups with laws

$\mathsf{pk} : s_0 \in S$



$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$\longrightarrow$

$\longleftarrow$

$\longrightarrow$

$\vdots$

$\vdots$

This can be generalised to the multi-variable case!

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1} g^{a_1} \dots h^{b_k} g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1} h^{d_1} \dots g^{c_\ell} h^{d_\ell}$

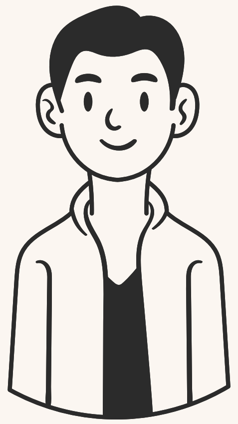
Key exchange protocol for actions of groups with laws

Alice

$g \xleftarrow{\$} G$

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$



Bob

$h \xleftarrow{\$} G$

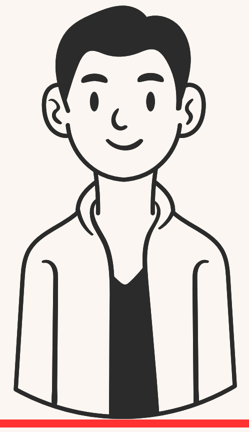
$$\longrightarrow$$
$$\longleftarrow$$
$$\longrightarrow$$
$$\vdots$$
$$\vdots$$

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1} g^{a_1} \dots h^{b_k} g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1} h^{d_1} \dots g^{c_\ell} h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

$\longrightarrow$

$\longleftarrow$

$\longrightarrow$

$\vdots$

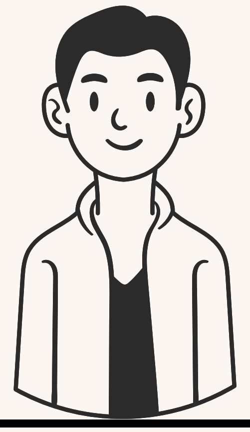
$\vdots$

key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1} g^{a_1} \dots h^{b_k} g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1} h^{d_1} \dots g^{c_\ell} h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice



Bob

$\mathsf{pk} : s_0 \in S$

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$
for any $x, y \in G$

length of a law $\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$

$g \xleftarrow{\$} G$

$h \xleftarrow{\$} G$

Multi-variable laws can be turned into 2-variable laws by a polynomial blow-up in length [Bradford-Thom, *TAMS*, 19]

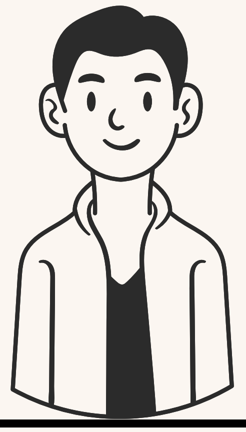
key : $s_k = t_k * g^{a_k} = s_0 * h^{b_1} g^{a_1} \dots h^{b_k} g^{a_k}$

key : $t'_\ell = s'_\ell * h^{d_\ell} = s_0 * g^{c_1} h^{d_1} \dots g^{c_\ell} h^{d_\ell}$

Key exchange protocol for actions of groups with laws

Alice

$\mathsf{pk} : s_0 \in S$



Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

length of a law

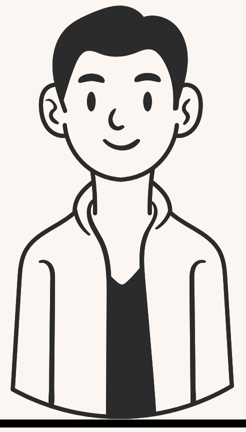
$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

Key exchange protocol for actions of groups with laws

Alice

pk : $s_0 \in S$



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$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

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1. One-way hardness (with multiple copies)

Key exchange protocol for actions of groups with laws

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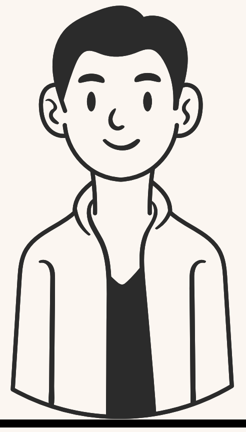

Bob

$$y^{b_1}x^{a_1}\ldots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\ldots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

length of a law

$$\longrightarrow \sum_{i=1}^{k}(a_i+b_i)+\sum_{i=1}^{\ell}(c_i+d_i)$$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
2. With a law whose length is as short as possible

Key exchange protocol for actions of groups with laws

Alice

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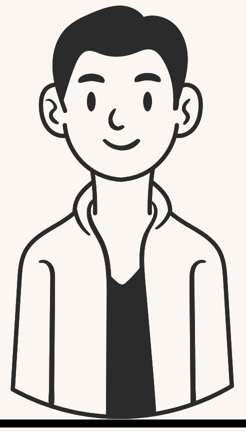

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$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

length of a law
$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

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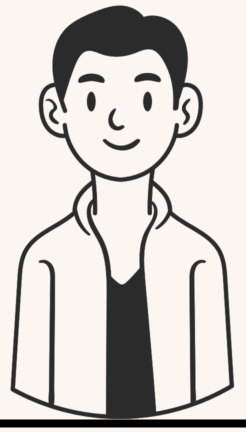
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Metabelian groups

Key exchange protocol for actions of groups with laws

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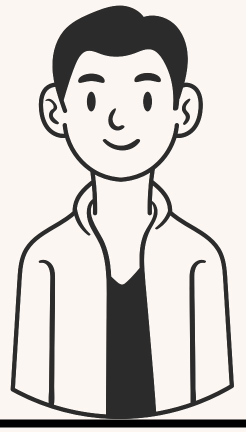

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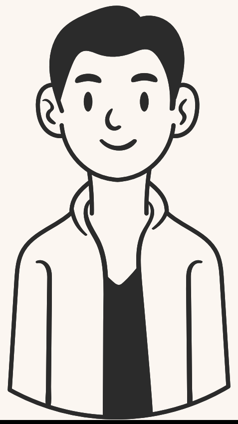

Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

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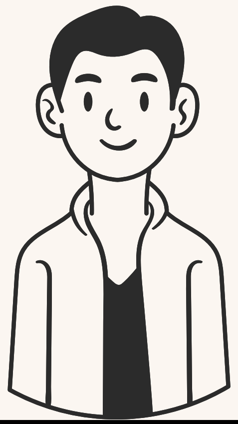

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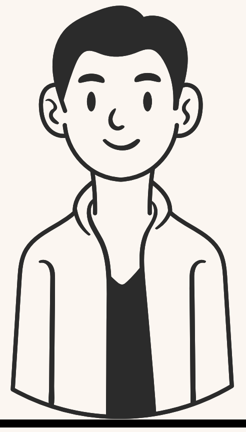

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length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
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Highly non-abelian groups,
e.g., general linear groups

Key exchange protocol for actions of groups with laws

Alice

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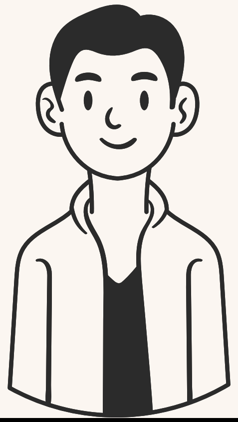

Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

length of a law

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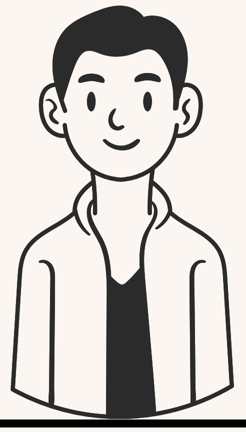

Bob

$$y^{b_1} x^{a_1} \dots y^{b_k} x^{a_k} = x^{c_1} y^{d_1} \dots x^{c_\ell} y^{d_\ell}$$

for any $x, y \in G$

length of a law $\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$

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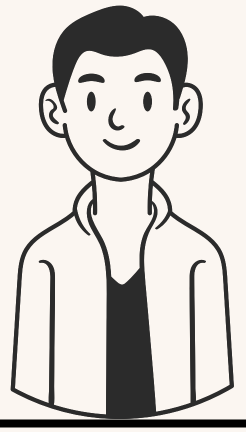

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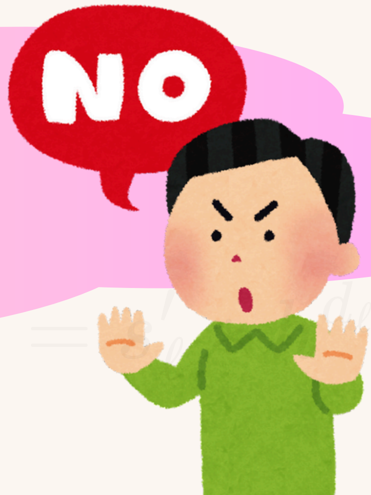

[Bradford-Thom, *JEMS*, 24]
The length could be exponentially long!

Key exchange protocol for actions of groups with laws

Alice

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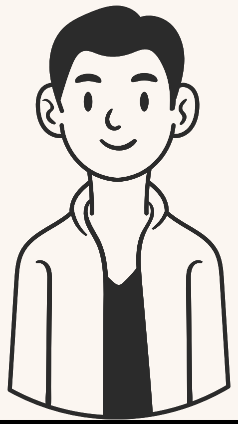

Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
2. With a law whose length is as short as possible

Highly non-abelian groups,
e.g., symmetric groups

Key exchange protocol for actions of groups with laws

Alice

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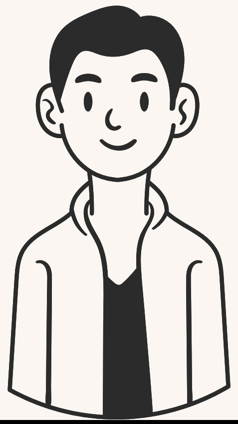

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length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies) ✔
2. With a law whose length is as short as possible

Highly non-abelian groups,
e.g., symmetric groups

Key exchange protocol for actions of groups with laws

Alice

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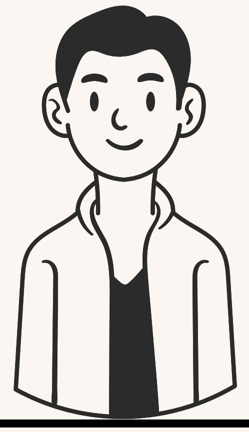

Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$

for any $x, y \in G$

length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

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Highly non-abelian groups,

e.g., symmetric groups

Key exchange protocol for actions of groups with laws

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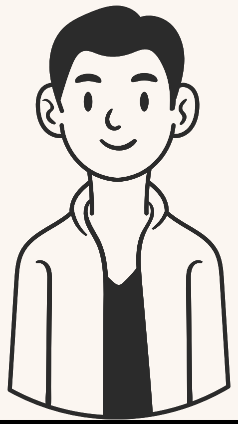

Bob

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for any $x, y \in G$

length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

What kind of (non-abelian) Group should we choose?

1. One-way hardness (with multiple copies)
2. There is a short law with high probability

Highly non-abelian groups,
e.g., symmetric groups

Key exchange protocol for actions of groups with laws

Alice

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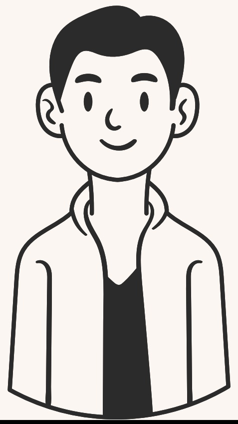

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length of a law

$$\sum_{i=1}^{k}(a_i + b_i) + \sum_{i=1}^{\ell}(c_i + d_i)$$

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i.e., $(xy)^n = \mathrm{id}$

Highly non-abelian groups,

e.g., symmetric groups

Key exchange protocol for actions of groups with laws


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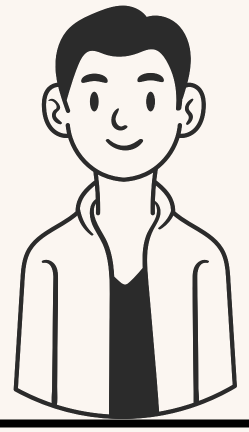
Bob

$$y^{b_1}x^{a_1}\dots y^{b_k}x^{a_k} = x^{c_1}y^{d_1}\dots x^{c_\ell}y^{d_\ell}$$
for any $x, y \in G$

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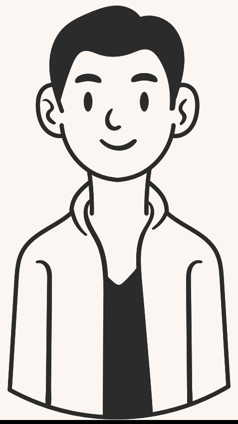

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Problem (Linear Code Equivalence)
For two generator matrices $C_1, C_2 \in \mathrm{M}(k \times n, \mathbb{F}_q)$, determine if there is $A \in \mathrm{GL}(k, \mathbb{F}_q)$ and $M \in \mathrm{Mon}(n, \mathbb{F}_q)$ such that $C_1 = AC_2M$.

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- We also carry out Magma experiments to support the hardness.

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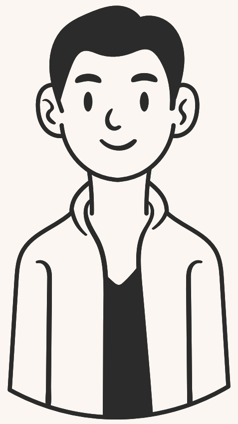

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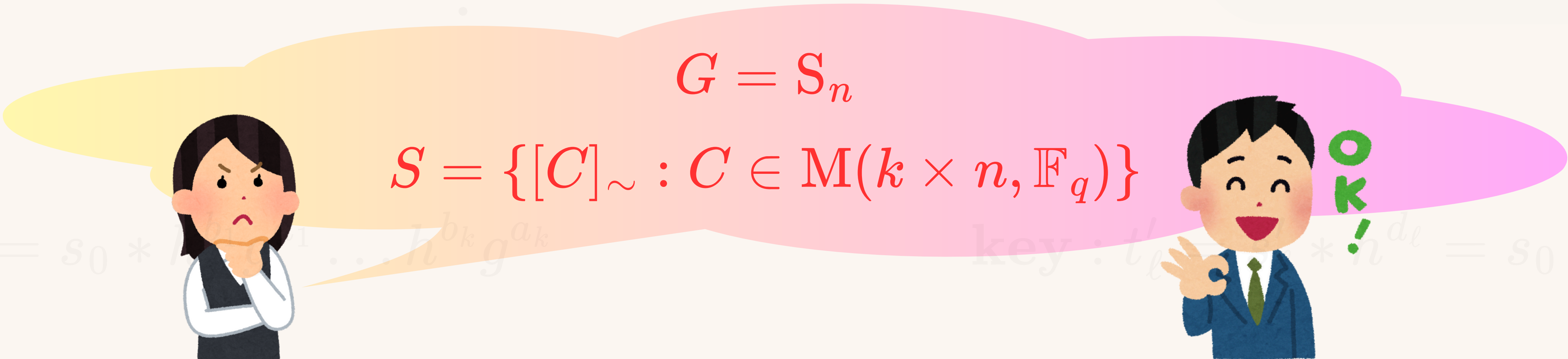

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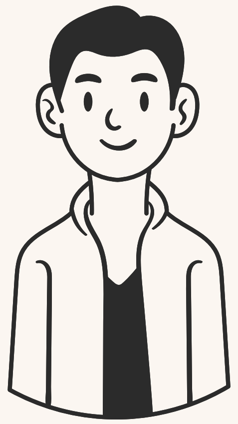

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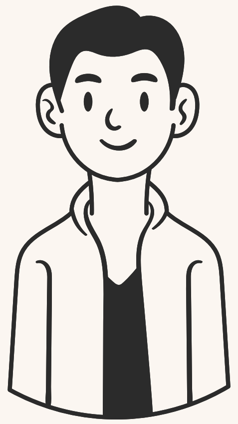

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Question and Answer

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