Faster isomorphism testing of p-groups of Frattini class-2

Speaker: Chuanqi Zhang

Joint with Gábor Ivanyos, Euan Mendoza, Youming Qiao, and Xiaorui Sun

Centre for Quantum Software and Information
University of Technology Sydney, Australia

Theory of Computing Seminar at the University of Wisconsin-Madison

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This work has been accepted to FOCS 2024 and invited to the SICOMP special issue.

Outline

- Background of finite group isomorphism.
- From p-group isomorphism to 3-tensor isomorphism.
- Our main results and an overview of our algorithm.
- Summary and further directions.

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Group Isomorphism Problem

Problem (GROUP ISO)

Given the multiplication tables of two groups of finite order N, determine whether they are isomorphic.

- First algorithm: $N^{\log N + O(1)}$ time attributed to Tarjan [Miller'78]

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Given the multiplication tables of two groups of finite order N, determine whether they are isomorphic.

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$+$	0	1
0	0	1
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$\cong$

S_2

$\circ$	e	s
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Group Isomorphism vs. Graph Isomorphism

Similarity

- Both are extensively studied since 1970s.
- Both are neither known to be in P nor to be NP-complete.
- Best algorithms for both problems have a quasi-polynomial running time.

Differences

	Group Iso	Graph Iso
Natural bound	$N^{\log N+O(1)}$ [Miller'78]	
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Through the lens of GROUP ISO testing

- Theoretical computer science: complexity in the worst case

- Computational group theory: practical algorithms (as in Magma or GAP)

- Cryptography: protocols based on isomorphism problems

All of these areas can give us good motivation to study GROUP ISO testing ;)

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Recent breakthrough in Group Iso of a special class

Theorem (Sun'23)

Given two p-groups of class-2 and exponent p of order N, there exists an algorithm in time $N^{\tilde{O}((\log N)^{5/6})}$ *to decide whether they are isomorphic.*

Why this class of groups?

1. The isomorphism testing between p-groups of class-2 and exponent p is a major bottleneck for Group Iso.

Definition (p-groups of class-2 and exponent p)

For prime p, we say a p-group G is of *class-2 and exponent p*, if

- every $g \in G$ satisfies that $g^p = \mathrm{id}$, and
- $[G, [G, G]]$ only contains the identity element.

2. The isomorphism testing between p-groups of class-2 and exponent p can reduce to the isomorphism testing between two 3-tensors.

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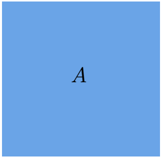
Tensor Isomorphism Problem

- Tensors are multi-way arrays, e.g., 2-tensor $A = (a_{i,j})_n$ is an $n \times n$ matrix:



- Similarly, 3-tensors are arrays with 3 indices, like a cube with matrix slices:

- More generally, we can define d-tensors, but d-Tensor Iso is as hard as 3-Tensor Iso. [Grochow-Qiao'23]

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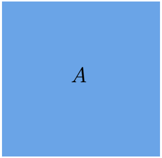

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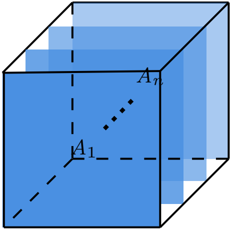

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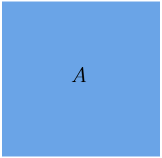

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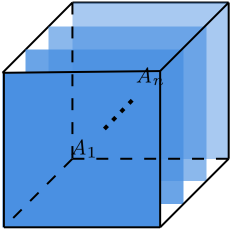

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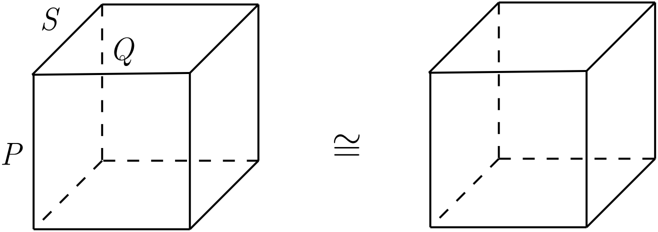
Isomorphism for 3-tensors under three invertible matrices P, Q, and S:



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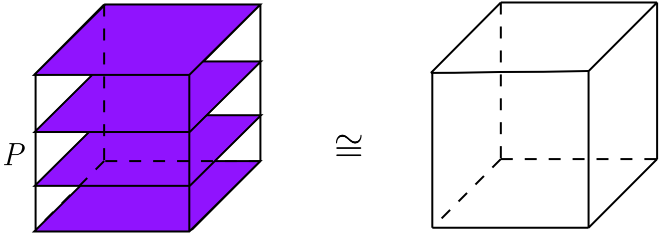

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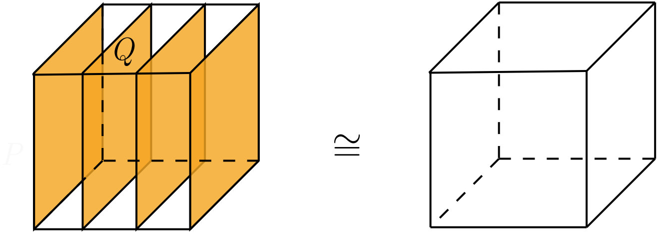

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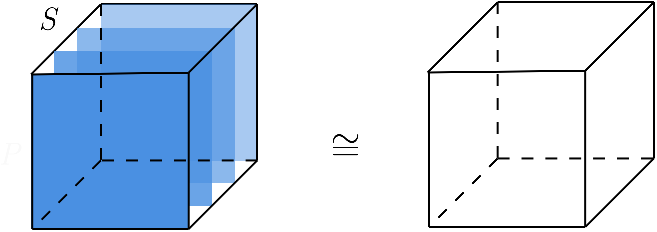

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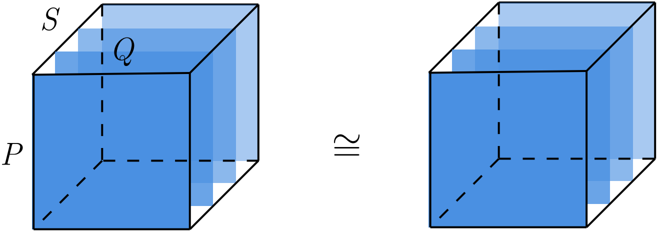

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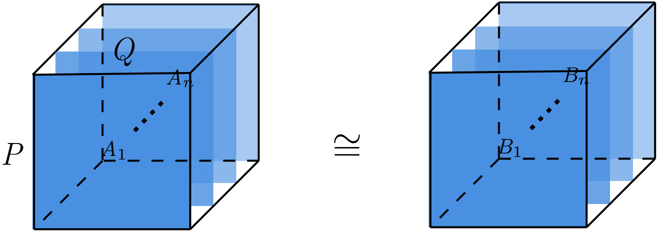

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Definition (Linear span of matrices)

Let $\{B_i : i \in [n]\}$ be a set of matrices over $\mathbb{F}_q$. Then

$$\mathrm{span}\{B_i : i \in [n]\} := \{\textstyle\sum_{i=1}^{n} c_i B_i : c_i \in \mathbb{F}_q\}.$$

Problem (Equivalence testing of 3-tensors)

Given two $n \times n \times n$ tensors over $\mathbb{F}_q$ whose frontal slices are $\{A_i : i \in [n]\}$ and $\{B_i : i \in [n]\}$, respectively. Determine if they are equivalent, i.e., if there exist two invertible matrices P and Q such that

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Problem (Congruence testing of 3-tensors)

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From p-GROUP ISO to 3-TENSOR ISO


p-groups of class-2 and exponent p with order-N

Theorem (Baer's correspondence)

Two p-groups of class-2 and exponent p are isomorphic if and only if their associated skew-symmetric 3-tensors over $\mathbb{F}_p$ are congruent.

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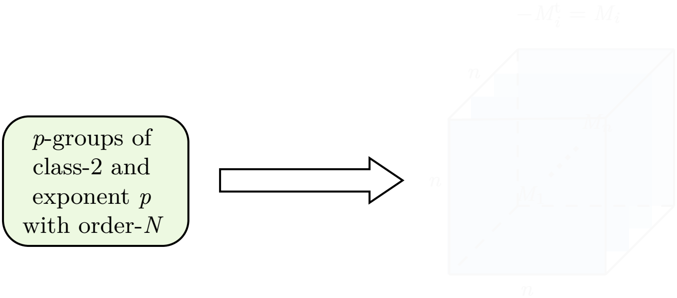

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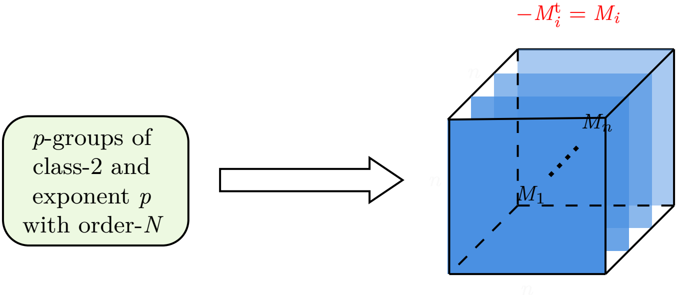

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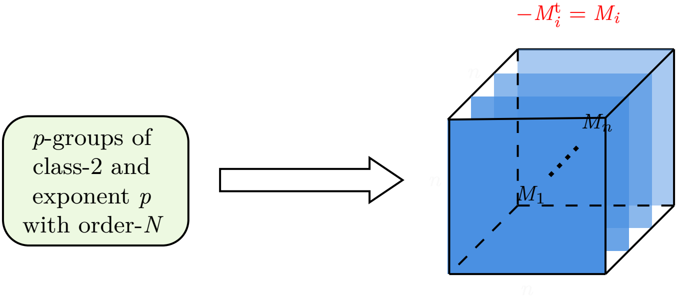

p-groups of
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$-M_i^{\mathrm{t}} = M_i$

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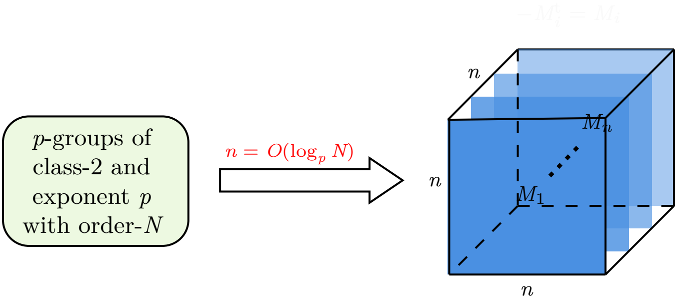

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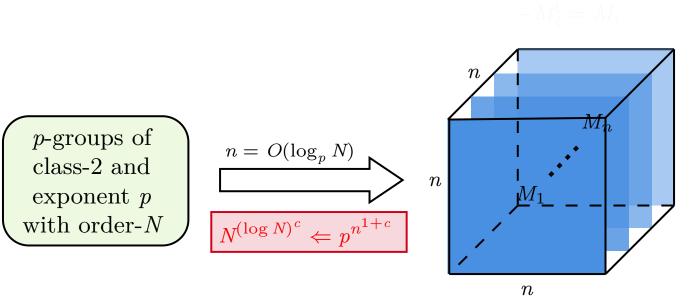

p-groups of class-2 and exponent p with order-N

$n = O(\log_p N)$

$N^{(\log N)^c} \Leftarrow p^{n^{1+c}}$

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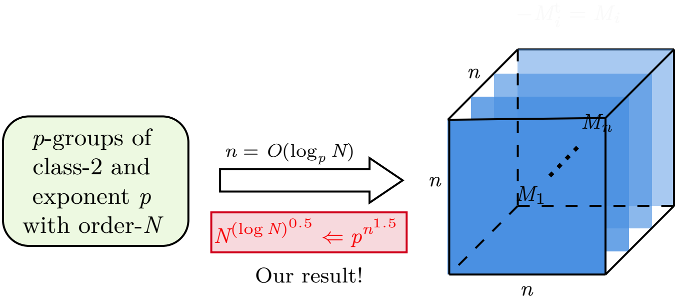

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Our result!

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Previous work and our main result

- Natural upper bound: $q^{O(n^2)}$ (known since at least 1970's)
- Sun's breakthrough: $q^{O(n^{1.8}\cdot\log q)}$ [Sun'23]
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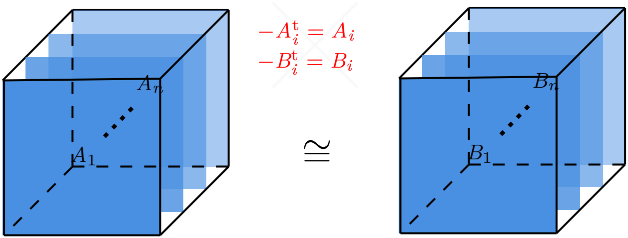

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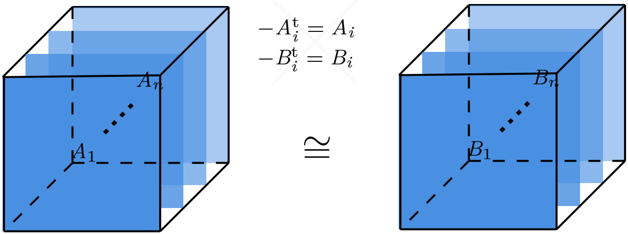

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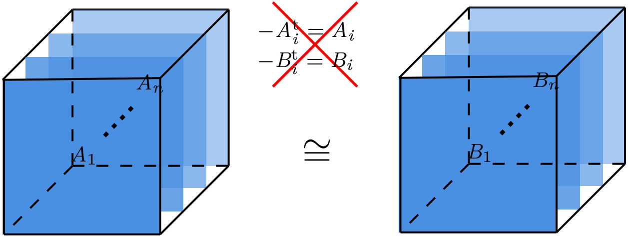

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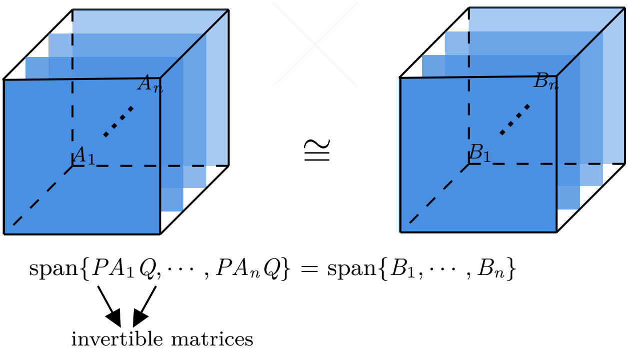

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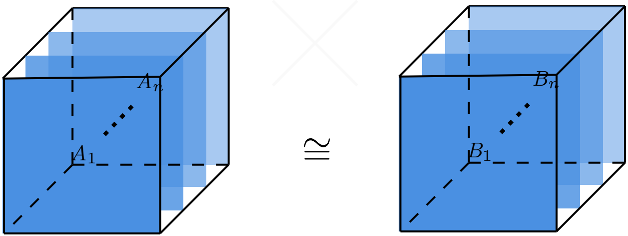

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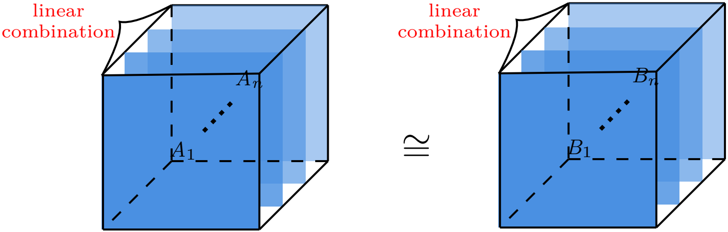

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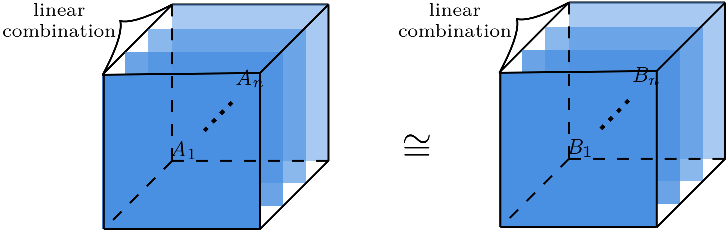

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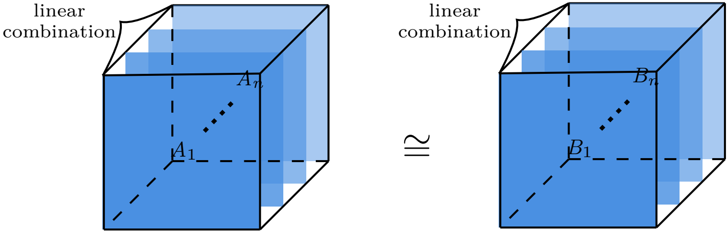

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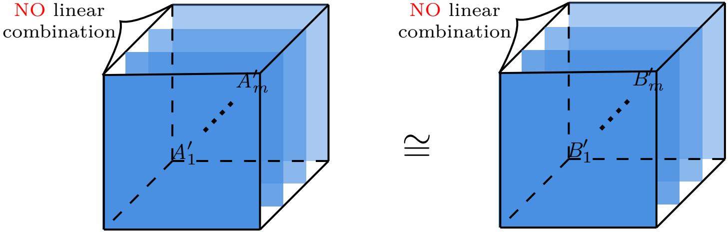

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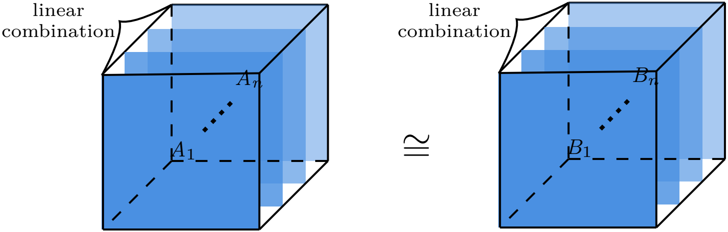

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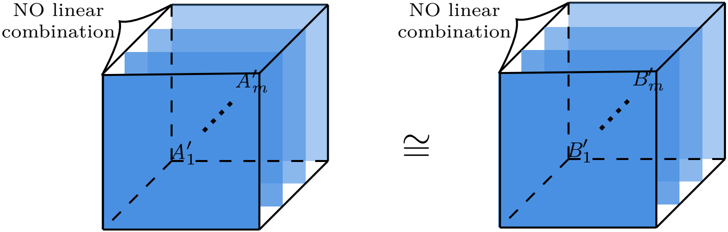

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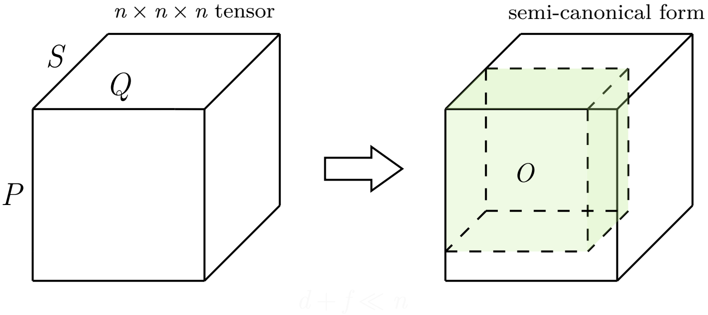

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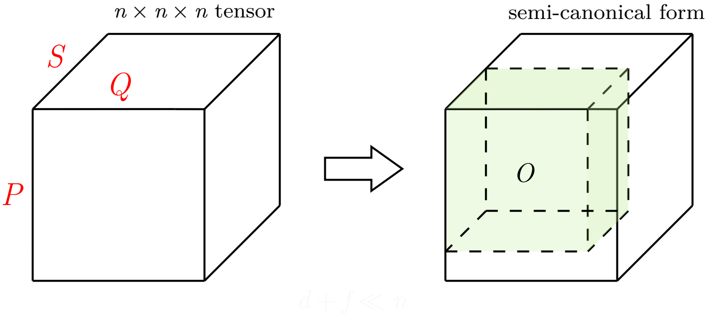

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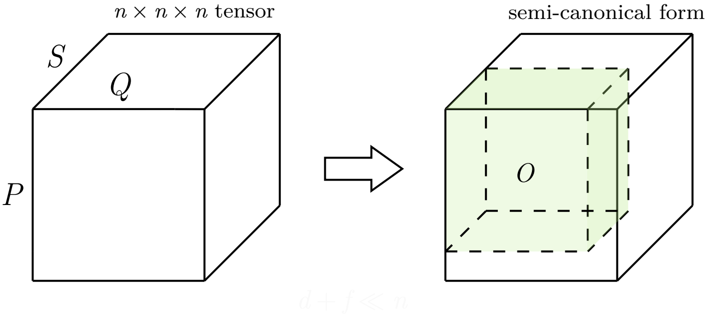

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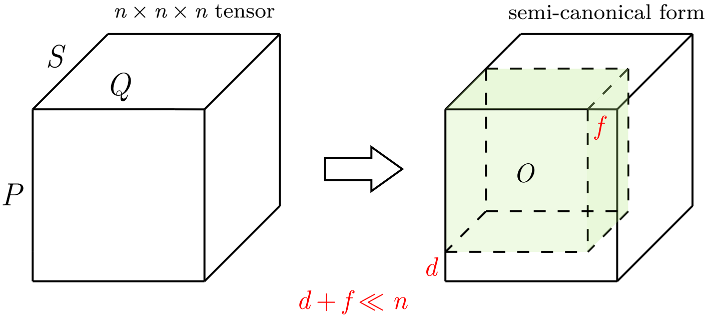

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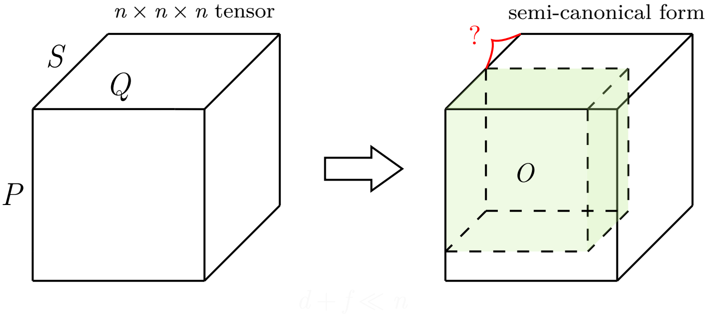

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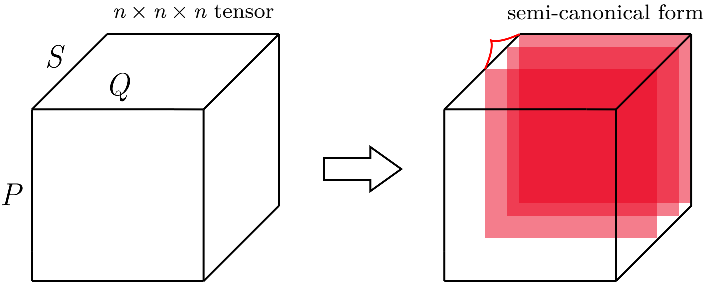

$d + f \ll n$

Two key techniques:

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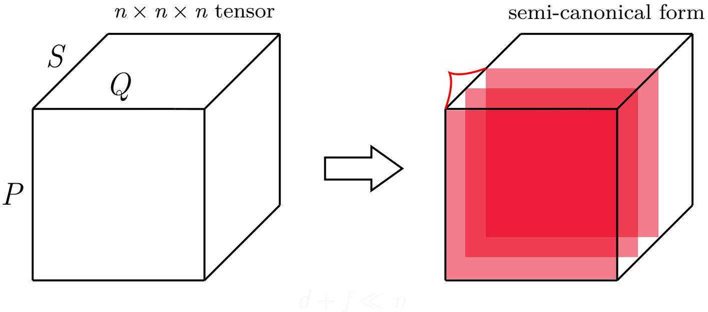

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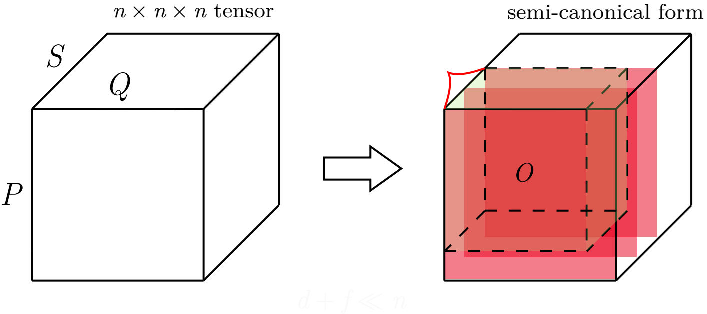

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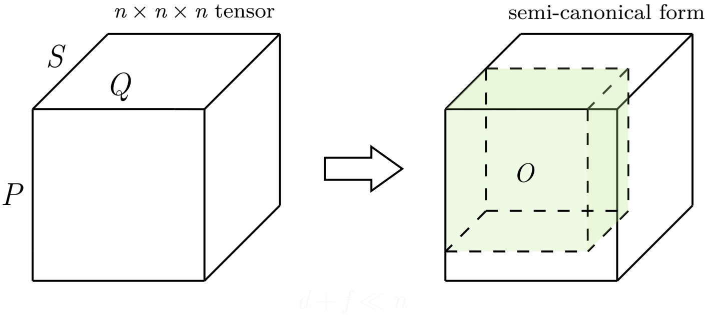

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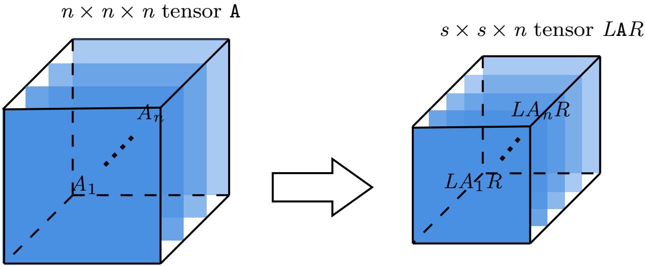

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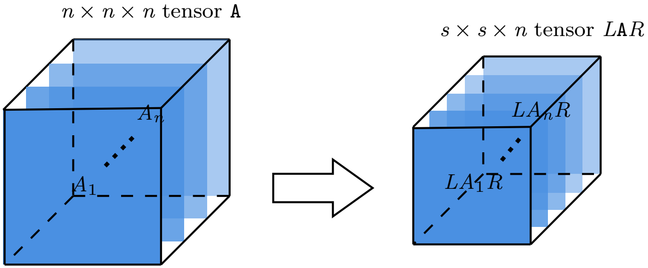

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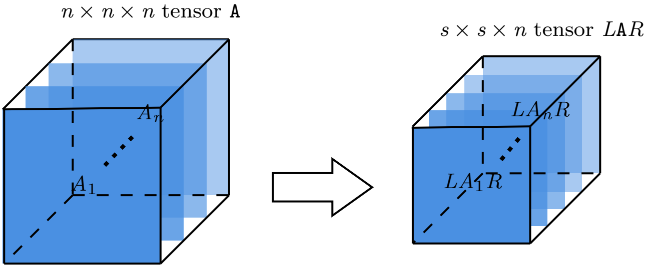

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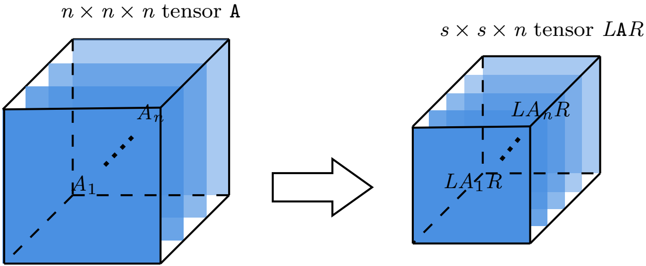

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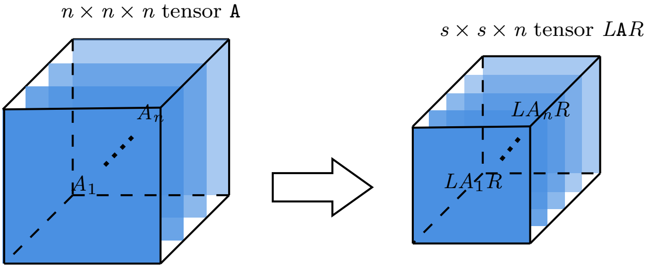

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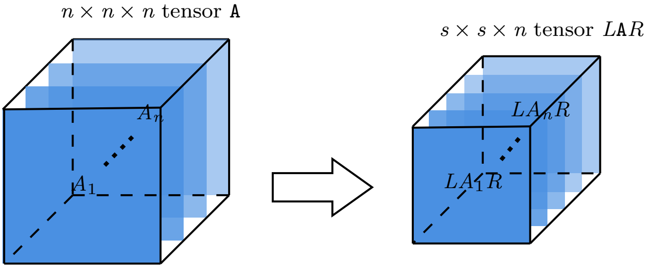

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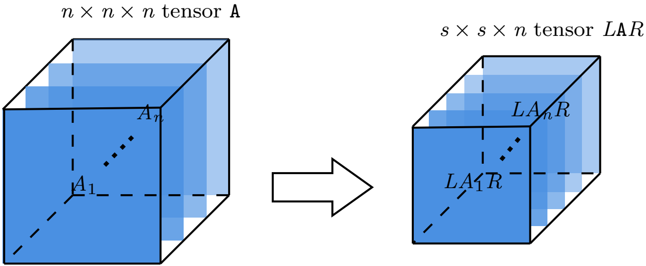

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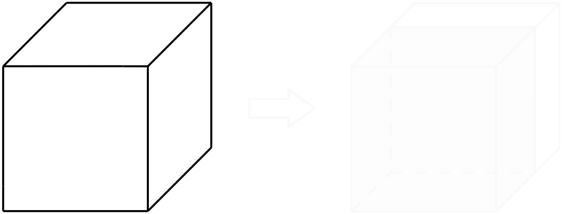
Technique 1: refine the frontal slices



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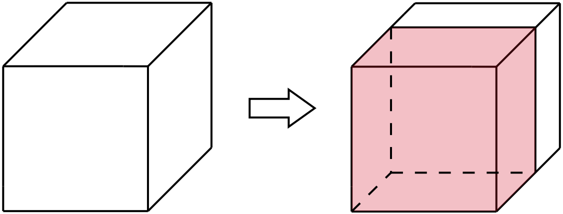

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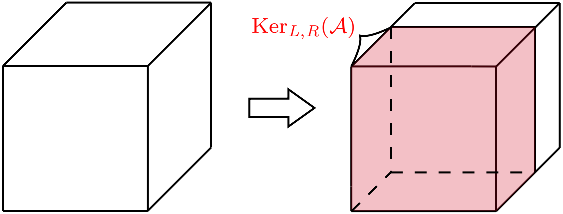

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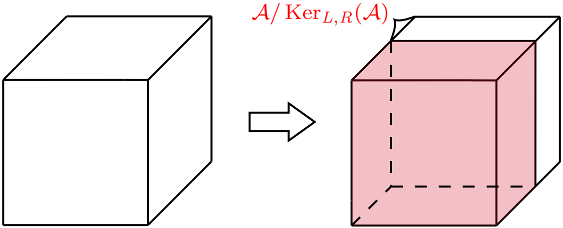

- Given a 3-tensor `A` whose frontal slices span $\mathcal{A} \leq \mathrm{M}(n, \mathbb{F}_q)$.
- Basic idea: sort the basis matrices (subject to choices of L, R) such that
  - the first ones span $\operatorname{Ker}_{L,R}(\mathcal{A}) := \operatorname{span}\{A \in \mathcal{A} \mid LAR = 0\}$, and
  - the remaining ones form a canonical basis of the quotient space $\mathcal{A}/\operatorname{Ker}_{L,R}(\mathcal{A})$.
- Advantage: $\operatorname{Ker}_{L,R}(\mathcal{A})$ is a low-rank subspace with a high probability.

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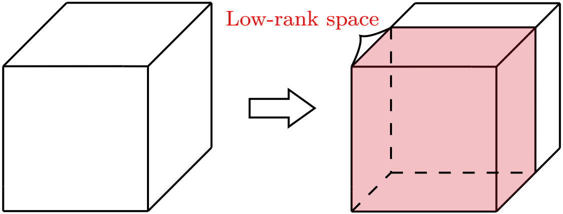

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Technique 1: refine the frontal slices of a 3-tensor

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Lemma (Ivanyos-Mendoza-Qiao-Sun-Zhang'24)

Let $\mathcal{A} \leq \mathrm{M}(n, \mathbb{F}_q)$ *be a matrix subspace of dimension* n. *Then with at least probability of* $1 - \frac{1}{q^r}$, $\mathrm{Ker}_{L,R}(\mathcal{A})$ *consists of matrices of rank* $\leq r$ *for uniformly randomly sampled* $L \in \mathrm{M}(s \times n, \mathbb{F}_q)$ *and* $R \in \mathrm{M}(n \times s, \mathbb{F}_q)$.

Why is this an advantage?

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$$s = \lceil 3 \cdot \max\{\frac{n}{r}, r\} \rceil.$$

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Again, to find L', R' such that `B` is refined correspondingly to `A`, we still need to enumerate all L', R' in the same size, which costs $q^{O(ns)}$. Why is this an advantage?

Technique 1: refine the frontal slices of a 3-tensor

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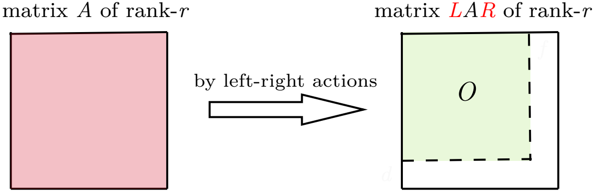
A trivial case: characterize a low-rank matrix



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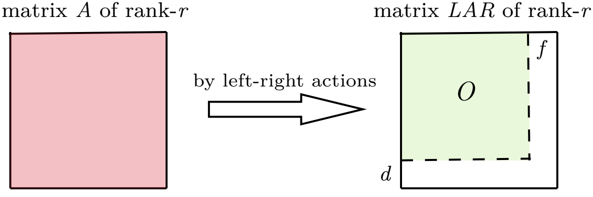

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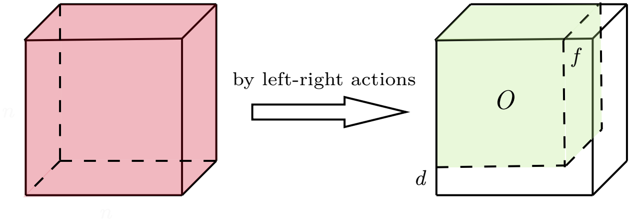
Technique 2: characterize a low-rank matrix subspace



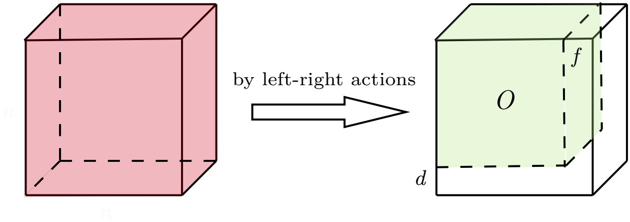
$$d + f = O(r)$$

over field of order $\geq r+1$

[Flanders'62]

Technique 2: characterize a low-rank matrix subspace

3-tensor bounded by a low rank r Equivalent 3-tensor after characterization



$$d + f = O(r^2)$$

[Sun'23]

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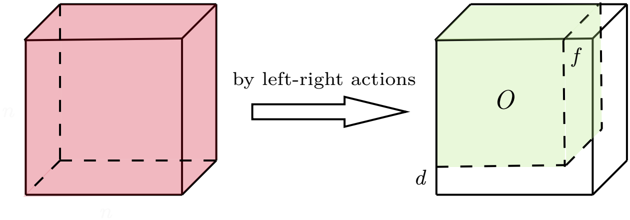

3-tensor bounded by a low rank r

by left-right actions

Equivalent 3-tensor after characterization

$$d + f = O(r \log r)$$

[Ivanyos-Mendoza-Qiao-Sun-Zhang'24]

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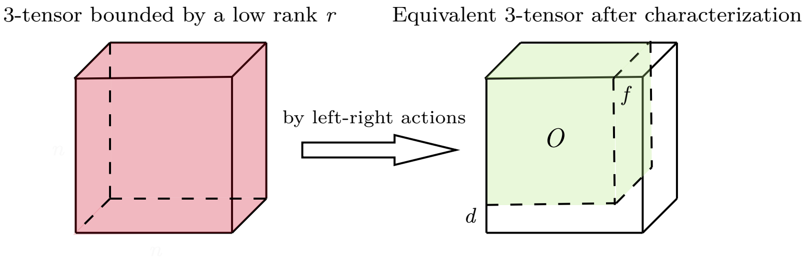

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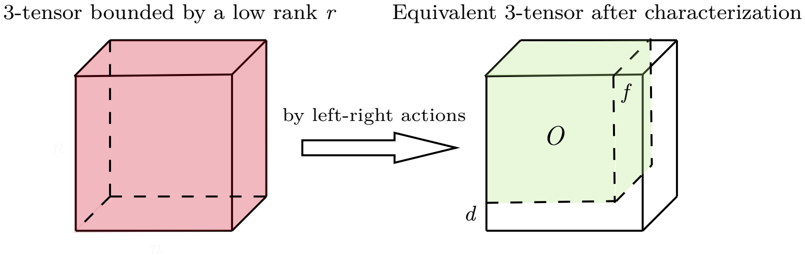

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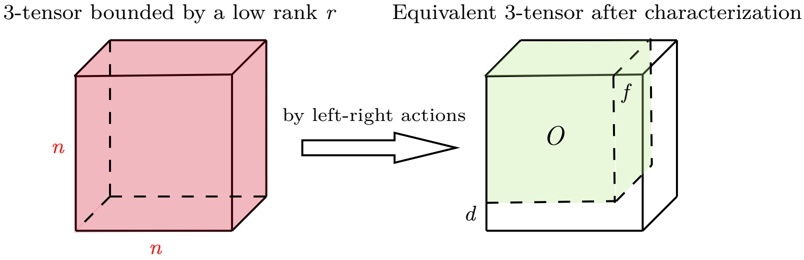

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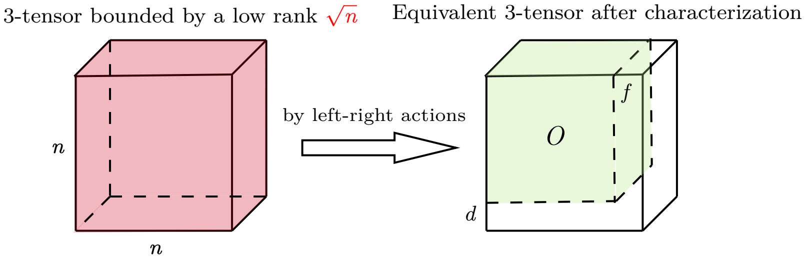

$$d + f = O(\sqrt{n} \log n)$$

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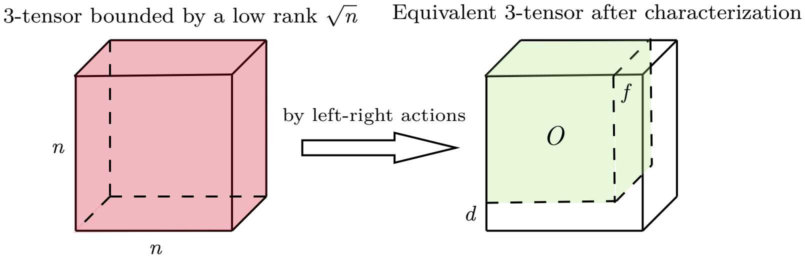

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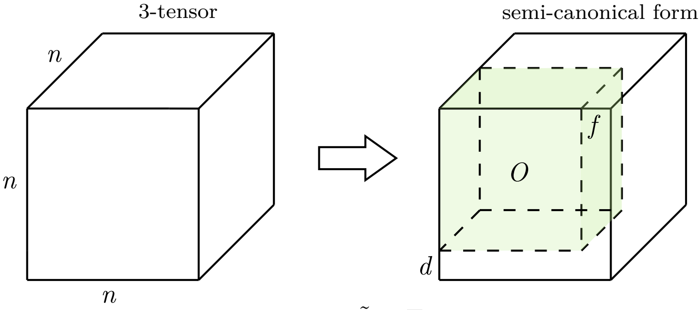
From semi-canonical 3-tensors to matrix tuples



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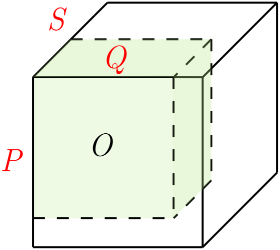

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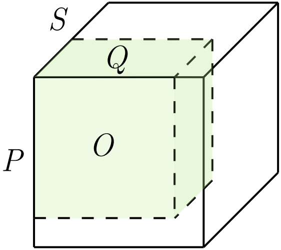
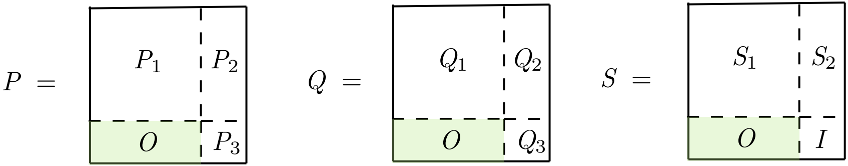


Upon enumeration which costs $q^{\tilde{O}(n^{1.5})}$,



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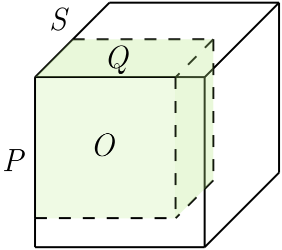

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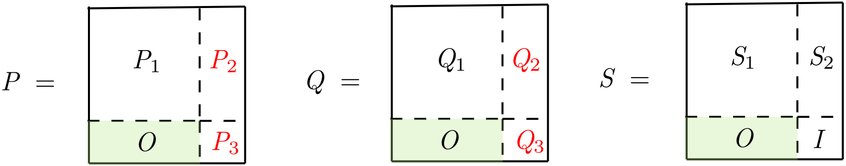

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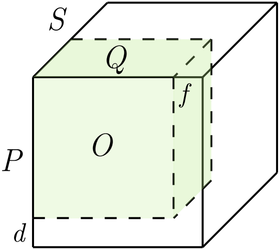

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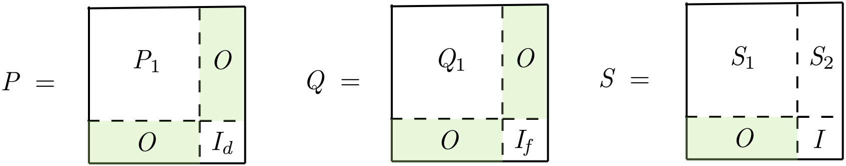

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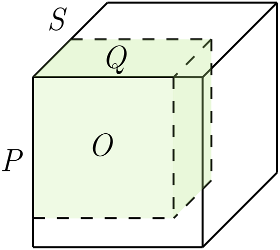

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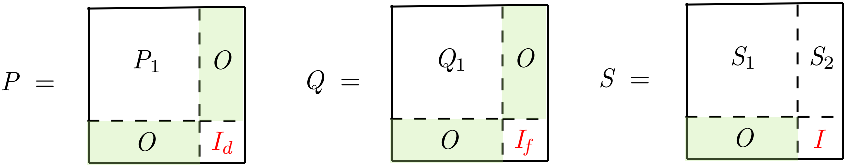
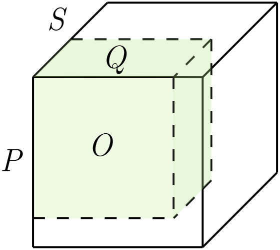
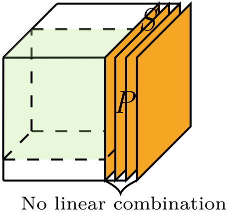
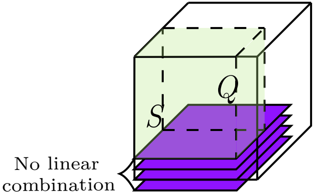


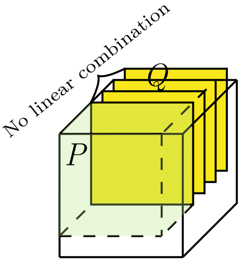
$$P = \begin{pmatrix} P_1 & O \\ O & I_d \end{pmatrix} \qquad Q = \begin{pmatrix} Q_1 & O \\ O & I_f \end{pmatrix} \qquad S = \begin{pmatrix} S_1 & S_2 \\ O & I \end{pmatrix}$$

From semi-canonical 3-tensors to matrix tuples









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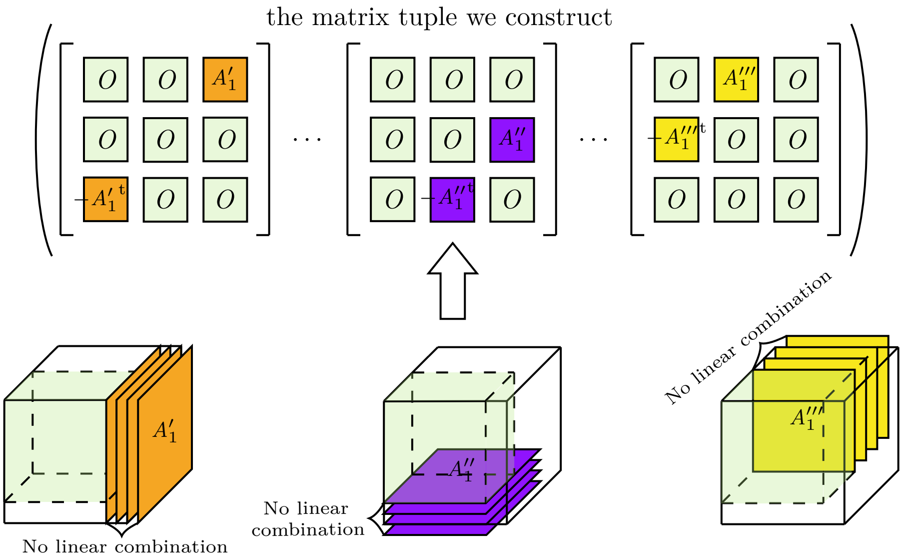
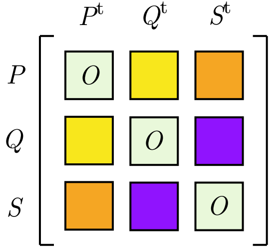
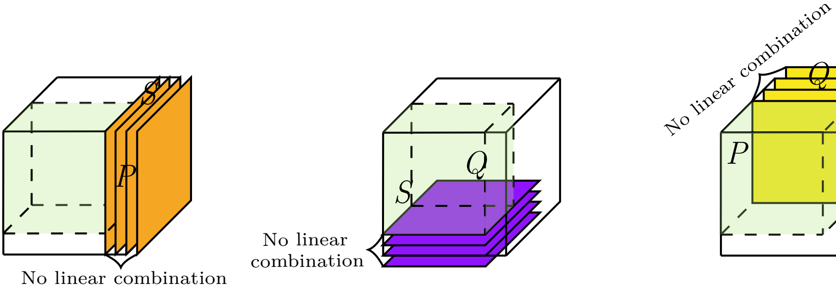


Some colourful slices may be transposed appropriately to match the action matrices.

From semi-canonical 3-tensors to matrix tuples



$$T = \begin{bmatrix} P & & \\ & Q & \\ & & S \end{bmatrix}$$



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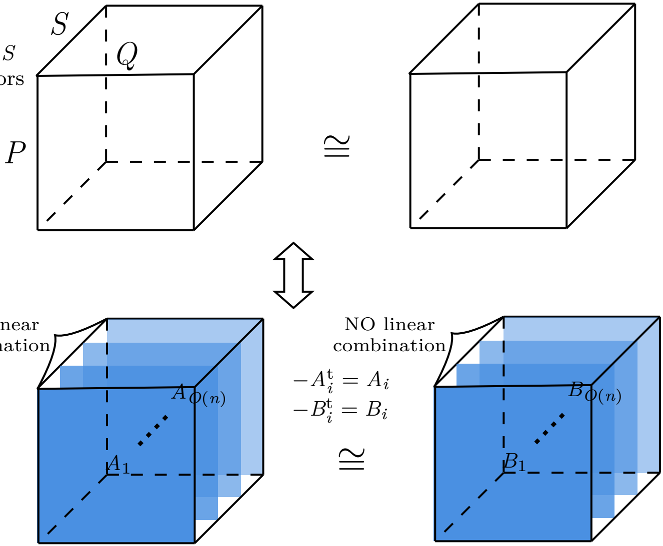
From 3-TENSOR ISO to (skew-symmetric) TUPLE ISO



∃ invertible matrices P, Q, S s.t. two 3-tensors $\cong$

$\Updownarrow$

$$-A_i^{\mathrm{t}} = A_i$$
$$-B_i^{\mathrm{t}} = B_i$$

$\cong$

∃ an invertible matrix T s.t. $(T^{\mathrm{t}} A_1 T, \cdots, T^{\mathrm{t}} A_{O(n)} T) = (B_1, \cdots, B_{O(n)})$

T is conditioned in a special form, but it is still reducible to the general problem.

Wrap-up of all the results

Theorem (Ivanyos-Qiao'19)

Given two skew-symmetric matrix tuples over $\mathbb{F}_q$, there exists a polynomial-time algorithm that decides whether they are congruent.

Theorem (Ivanyos-Mendoza-Qiao-Sun-Zhang'24)

Given two $n \times n \times n$ tensors over $\mathbb{F}_q$, there exists an algorithm in time $q^{\tilde{O}(n^{1.5})}$ that decides whether they are equivalent.

Theorem (Ivanyos-Mendoza-Qiao-Sun-Zhang'24)

Given two p-groups of Frattini class-2 of order N, there exists an algorithm in time $N^{\tilde{O}((\log N)^{1/2})}$ to decide whether they are isomorphic.

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Question and Answer

Thank you so much!

Please find the paper and slides available on my webpage:

https://www.chuanqizhang.com