Connections between graphs and matrix spaces

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Outline

- A quick introduction about matrix spaces.
- Starting point: Existence of perfect matchings $\iff$ Singularity
- A general framework of such connections.
- Another example: Acyclicity $\iff$ Nilpotency
- More results with implication to quantum information theory.

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What is a matrix space?

- A matrix space is a linear space spanned by matrices.
- Let $\mathrm{M}(n, \mathbb{F})$ denote the linear space of $n \times n$ matrices over a field $\mathbb{F}$. Then a linear subspace $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is called a matrix space.
- Specify a basis $M_1, \ldots, M_d$ for $\mathcal{S}$.
- $\mathcal{S}$ is the set of all linear combinations of $M_1, \ldots, M_d$.
- $\mathcal{S}$ corresponds to the symbolic matrix $x_1 M_1 + \cdots + x_d M_d$, whose entries are linear forms in the variables $x_1, \ldots, x_d$, e.g.,

$$x_1 \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 & 2x_1 \\ 2x_2 - x_1 & -2x_2 \end{pmatrix}.$$

- $\mathcal{S}$ corresponds to a superoperator $\Phi(X) = \sum_{i=1}^{d} M_i X M_i^*$.

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From graphs to matrix spaces

- For $n \in \mathbb{N}$, $[n] := \{1, 2, \ldots, n\}$.
- For $(i, j) \in [n] \times [n]$, let $\mathrm{E}_{i,j}$ be the elementary matrix in $\mathrm{M}(n, \mathbb{F})$ where the (i, j)th entry is 1, and the remaining entries are 0. For example,

$$\mathrm{E}_{2,3} \in \mathrm{M}(3, \mathbb{F}) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

- For a bipartite graph $G = ([n] \cup [n], E)$ or a directed graph $G = ([n], E)$, the adjacency matrix is

$$A_G := \sum_{(i,j) \in E} \mathrm{E}_{i,j}.$$

- The graphical matrix space (over $\mathbb{F}$) corresponding to G is

$$\mathcal{S}_G := \mathrm{span}\{\mathrm{E}_{i,j} \mid (i, j) \in E\}.$$

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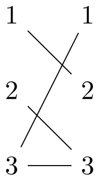
Our starting point

Bipartite Graph G



Graphical Matrix Space $\mathcal{S}_G$

$$\begin{pmatrix} 0 & x_1 & 0 \\ 0 & 0 & x_2 \\ x_3 & 0 & x_4 \end{pmatrix}$$

Theorem (Tutte'1947, Edmonds'1967, Lovász'1979)

G has a perfect matching iff $\mathcal{S}_G$ has some invertible matrices.

Proof sketch.

($\Rightarrow$) Take the matrix supporting on a perfect matching. This would yield an invertible matrix.

($\Leftarrow$) Take the symbolic matrix of $\mathcal{S}_G$. Existing invertible matrices implies the determinant polynomial $\sum_\sigma \mathrm{sgn}(\sigma) \prod_{i=1}^n x_{i,\sigma_i} \neq 0$ and thereby $\prod_{i=1}^n x_{i,\sigma_i} \neq 0$ for some σ. Then the edge set $\{(i, \sigma_i) : i \in [n]\}$ gives a perfect matching. $\square$

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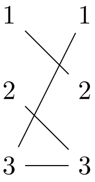

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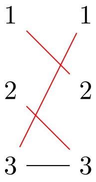

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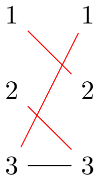

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Further observation

Fact (about perfect matching)

If a bipartite graph $G = ([n] \times [n], E)$ *doesn't contain any perfect matching, then* $|E| \leq n(n-1)$.

Problem

If $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ *doesn't contain any invertible matrix, how large can* $\mathcal{S}$ *be?*

- $\dim(\mathcal{S}) = n(n-1)$ is possible by this example,

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Theorem (Dieudonné'1948, Flanders'1962, Meshulam'1985)

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Correspondences between matrix spaces and graphs

- G has no perfect matching $\iff$ $\mathcal{S}_G$ has no invertible matrix
  - We call it a basic correspondence between G and $\mathcal{S}_G$.
- Max. size of such $G \subseteq K_{n,n}$ = Max. dim of such
  - Note that $\mathcal{S}$ doesn't have to be graphical.
- Max. size of such $G \subseteq H$ Max. dim of such $\mathcal{S} \leq \mathcal{S}_H$

Theorem (Li–Qiao–Wigderson–Wigderson–Zhang'2022)

For any bipartite graph H, the maximum size over all $G \subseteq H$ with no perfect matching = the maximum dim over all $\mathcal{S} \leq \mathcal{S}_H$ with no invertible matrix.

- This theorem generalizes Dieudonné's theorem.
- A combinatorial "explanation" of an algebraic property!
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Connection between acyclicity and nilpotency

- A graph is acyclic if it has no cycles.
- A matrix B is nilpotent, if $B^k = 0$ for some $k \in \mathbb{N}$.
- A directed graph G is acyclic iff its adjacency matrix A_G is nilpotent.
- Note that this doesn't hold over the field of order 2. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Fact (about directed acyclic graph)

If a directed graph $G = ([n], E)$ doesn't contain any cycles, then $|E| \leq \binom{n}{2}$.

Problem

If $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$ is nil, how large can $\mathcal{S}$ be?

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- This generalizes Gerstenhaber's theorem.
- We adapt de Seguins Pazzis's proof [dSP13] of Gerstenhaber's theorem to prove the $\geq$ direction.
- Corollary: Given $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$, it is NP-hard to determine the maximum dimension of nil subspace of $\mathcal{S}$.

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Summary

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for a graph-theoretic property P and a linear-algebraic property Q.

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- We also have other results: strong-connectivity and irreducibility, isomorphism and conjugacy/congruence...
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- Embed a d-regular graph G to a quantum channel $\Phi_G : \mathrm{M}(n, \mathbb{C}) \to \mathrm{M}(n, \mathbb{C})$ [Bannink–Brïet–Labib–Maassen'2020].
- For every $X \in \mathrm{M}(n, \mathbb{C})$, the graphical quantum channel of G is defined as

$$\Phi_G(X) := \frac{1}{d} \sum_{(i,j) \in E} \mathrm{E}_{i,j} X \mathrm{E}_{i,j}^*.$$

Theorem (Bannink–Brïet–Labib–Maassen'2020, Proposition 3.7)

For any d-regular graph G, the spectral expansion of G equals the spectral expansion of Φ_G.

- Φ is irreducibly covariant, if there exists a compact group Γ and a continuous irreducible unitary representation $U : \Gamma \to \mathrm{U}(n)$ such that for any $g \in \Gamma$ and $X \in \mathrm{M}(n, \mathbb{C})$, we have $\Phi(U(g) X U(g)^*) = U(g) \Phi(X) U(g)^*$.

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Vertex-transitivity

- Let G be a directed graph. Let $\mathrm{Aut}(G)$ be the automorphism group of G.
- Recall that G is vertex-transitive, if $\mathrm{Aut}(G)$ is a transitive group.

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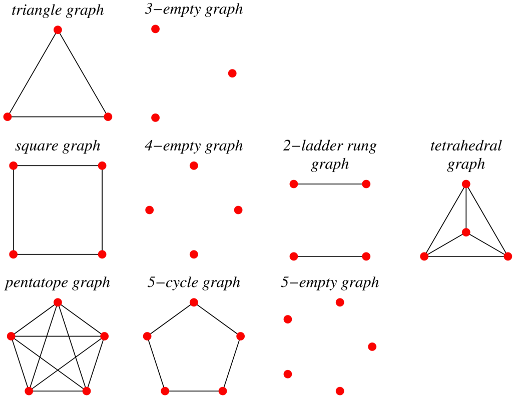

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Conjugacy/congruence irreducibility

- Let matrix group $\mathcal{G} \leq \mathrm{GL}(n, \mathbb{F})$ and $U \leq \mathbb{F}^n$.
- $\mathcal{G}$ is reducible if there exists a non-zero and proper U such that for any $A \in \mathcal{G}$, $A(U) \leq U$. Otherwise, we call $\mathcal{G}$ irreducible.
- In this case, U is called an invariant subspace.

- Let matrix space $\mathcal{S} \leq \mathrm{M}(n, \mathbb{F})$.
- Define $\mathrm{Conj}(\mathcal{S}) := \{T \in \mathrm{GL}(n, \mathbb{F}) \mid T\mathcal{S}T^{-1} = \mathcal{S}\} \leq \mathrm{GL}(n, \mathbb{F})$. We say that $\mathcal{S}$ is conjugacy irreducible, if $\mathrm{Conj}(\mathcal{S})$ is irreducible as a matrix group.
- Define $\mathrm{Cong}(\mathcal{S}) := \{T \in \mathrm{GL}(n, \mathbb{F}) \mid T\mathcal{S}T^t = \mathcal{S}\} \leq \mathrm{GL}(n, \mathbb{F})$. We say that $\mathcal{S}$ is congruence irreducible, if $\mathrm{Cong}(\mathcal{S})$ is irreducible as a matrix group.

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Implication to quantum information

Theorem (Li–Qiao–Wigderson–Wigderson–Zhang'2022)

Let G be a d-regular graph, and $\mathcal{S}_G$ and Φ_G be the graphical matrix space and quantum channel associated with G, respectively. Then the following are equivalent:

1. *G is vertex-transitive.*
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Theorem (Bannink–Brïet–Labib–Maassen'2020, Proposition 3.7)

For any d-regular graph G, the spectral expansion of G equals the spectral expansion of Φ_G.

- Inspired by this work in [BBLM20], we also investigated the linear-algebraic expanders generalized from graphs in the follow-up work[1].

Theorem (Li–Qiao–Wigderson–Wigderson–Zhang'2022)

For any undirected graph G, the vertex expansion of G equals the dimension expansion of $\mathbf{B}_G$.

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Some explanation

- Let G be an undirected graph.
- The vertex expansion of G is defined as

$$\mu(G) := \min_{\substack{W \subseteq [n] \\ 1 \leq |W| \leq \frac{n}{2}}} \frac{|\partial_{\text{out}}(W)|}{|W|},$$

where $\partial_{\text{out}}(W) := \{j \in [n] \setminus W : \exists\ i \in W, \text{ s.t. } \{i,j\} \in E\}$.

- Let $\mathbf{B} := (B_1, \ldots, B_m) \in \mathrm{M}(n, \mathbb{F})^m$ be a matrix tuple.
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- Let $\mathbf{B}_G := (\mathrm{E}_{i,j} : \{i,j\} \in E(G))$ be the graphical matrix tuple of G.
- Consider when V is a coordinate subspace, e.g, $V = \mathrm{span}\{e_2, e_3\}$.
- In this case, V is treated as a vertex subset, so $\dim(V)$ corresponds to $|W|$.
- Note that $\mathrm{E}_{i,j} e_j = e_i$, and $\mathrm{E}_{i,j} e_k = 0$ when $k \neq j$.
- In this case, $\dim(V + \mathbf{B}(V)) - \dim(V)$ is doing the same as $|\partial_{\mathrm{out}}(W)|$.

Theorem (Li–Qiao–Wigderson–Wigderson–Zhang'2022)

For any undirected graph G, $\mu(G) = \mu(\mathbf{B}_G)$.

- The proof is based on the ideas of [Dvir–Shpilka'11, Dvir–Wigderson'10].

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- In the classical setting, all three expansion are equivalent.
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Summary

- Basic correspondence: for any bipartite/directed graph G,

$$G \text{ satisfies } P \iff \mathcal{S}_G \text{ satisfies } Q$$

for a graph-theoretic property P and a linear-algebraic property Q.

- Inherited correspondence: for any bipartite/directed graph H,

$$\text{Max. size of } G \subseteq H \text{ satisfying } P = \text{Max. dim of } \mathcal{S} \leq \mathcal{S}_H \text{ satisfying } Q$$

- From a graph G, we can construct not only $\mathcal{S}_G$ but also Φ_G, $\mathbf{B}_G$, and even in other context, then establish the connection between their properties.

Question and Answer

Thank you so much!